Performance measurement and benchmarking

Contents of this lecture:

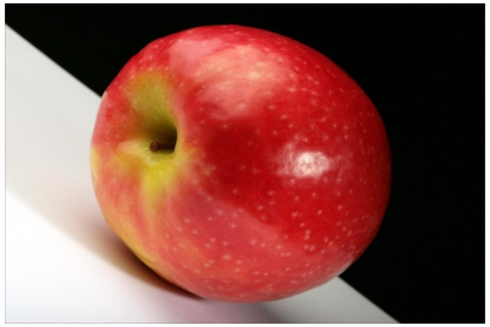
- Why measure performance?
- Why benchmarking?
- Benchmarking techniques
  - → DEA: Data Envelopment Analysis
  - → SFC: Stochastic Frontier Analysis

A benchmark

Benchmark: geological reference point
(altitude above sea level)

Performance measurement: from which perspective?

(source: C. Armado, R. Dyson, ESI-ORAHS 2005)



- What is performance?
- Sorting out the good from the bad.
  - But, *who* defines what is good or bad?
  - *Whose* interests are organizations answering to? (Fitzgerald and Storbeck, 2002)
  - Performance is socially constructed and means different things to different people (Wholey, 1996)

→ Stakeholder analysis

Benchmarking in healthcare

AD www.AD.nl Ziekenhuis top 100

ELSEVIER

PLEXUS MEDICAL GROUP

NFU NEDERLANDSE FEDERATIE VAN UNIVERSITAIR MEDISCHE CENTRA

NFU Benchmarking-OK

Flevoziekenhuis nummer 1 van Nederland

AD-hoofdredacteur Peter de Jonge overhandigt de trofee aan Jeltje Schraverus, voorzitter van de raad van bestuur.

ALMERE - Het Flevoziekenhuis in Almere is het best presterende ziekenhuis van het land. Dat blijkt uit de nieuwe AD Ziekenhuis top 100 die zaterdag verschijnt in het AD. Het Flevo steeg vorig jaar al spectaculair in de ranglijst naar nummer 29, maar weet de kwalititeitsslag die het op tal van medische terreinen boekte nu te verzilveren met een koppositie.

The four Es of healthcare

Performance = Efficiency?

NO!

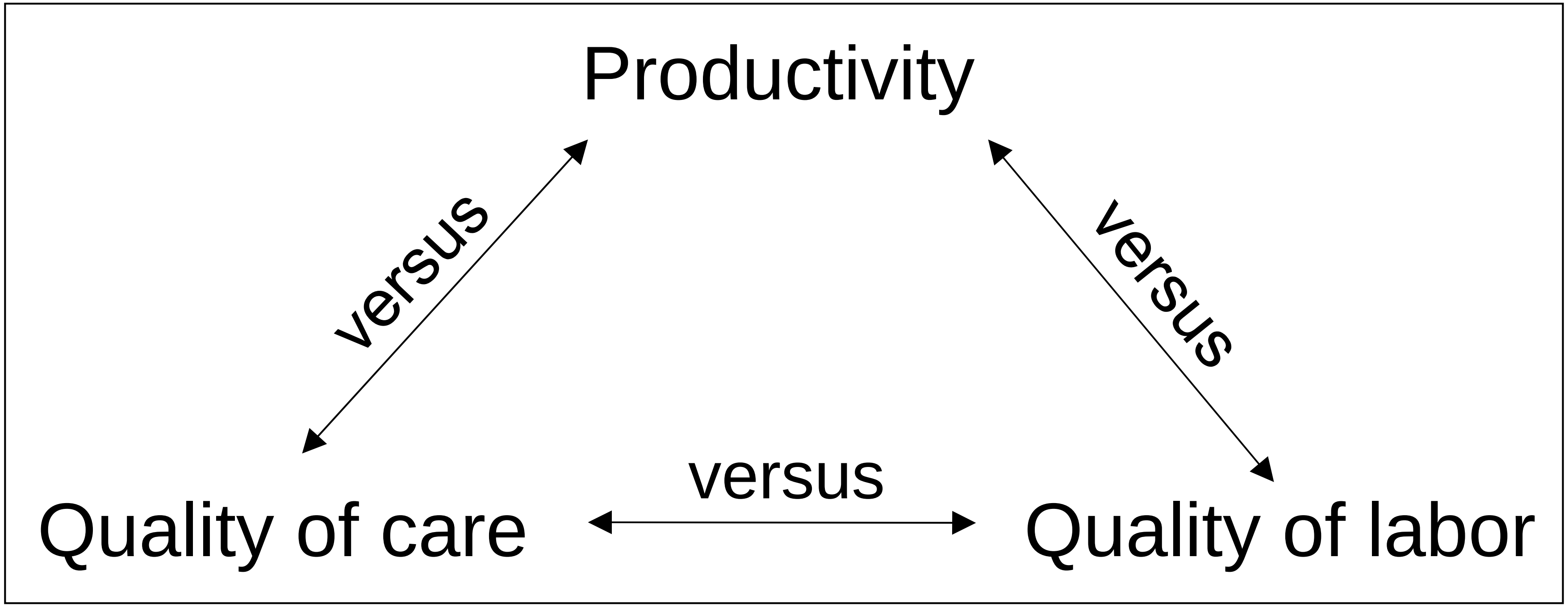
The three E-s:

Economy ↔ Efficiency ↔ Effectiveness

$$$ → Inputs → Outputs → Outcomes

And the fourth E: Equity!   (equity = fairness)

Multiple objectives



None of these objectives should be used individually

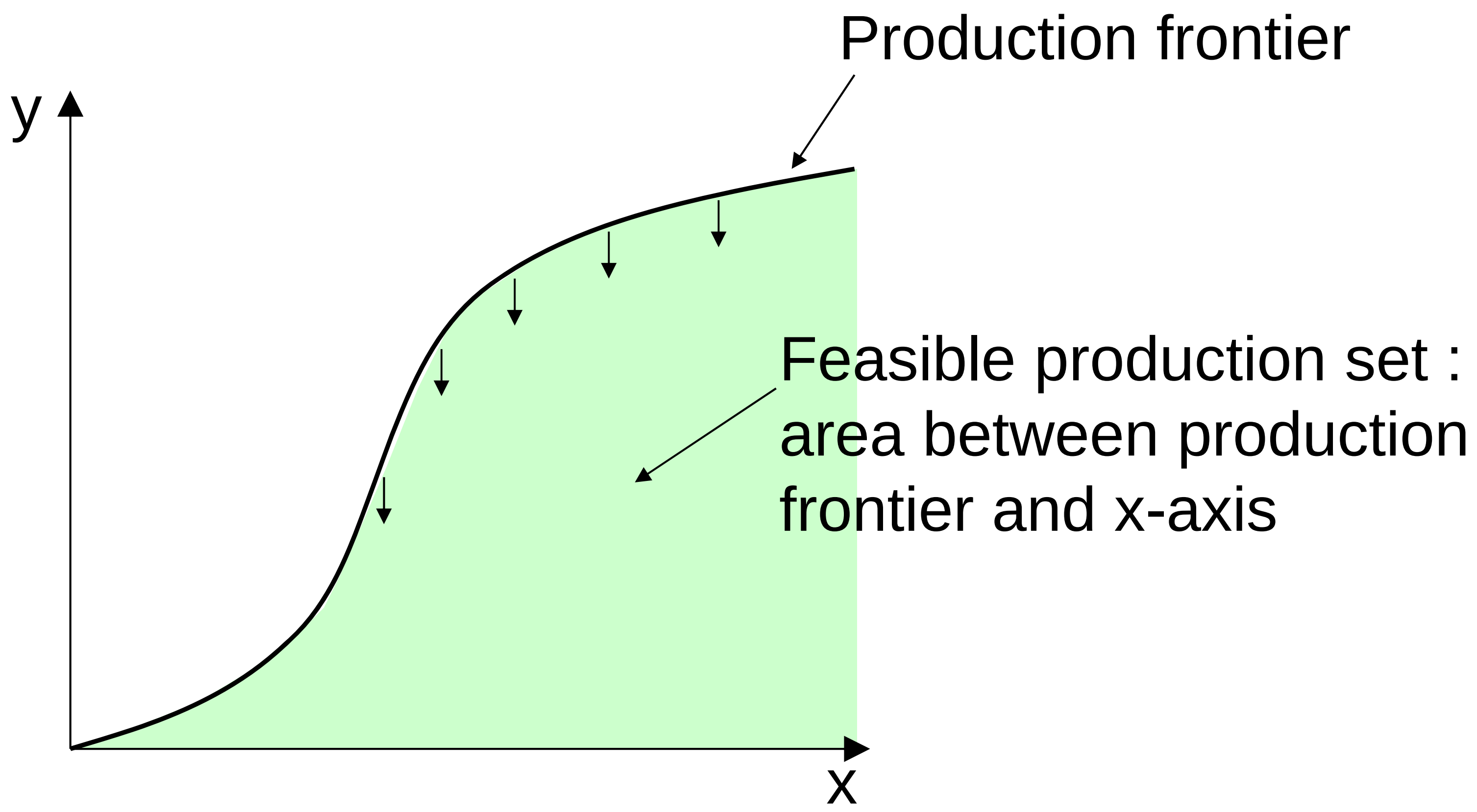
What is productivity?

Hospital A has 1200 beds, and handles 100,000 patients per year

- What is the productivity?
- What is the efficiency?
- Can the hospital improve these?
- How?

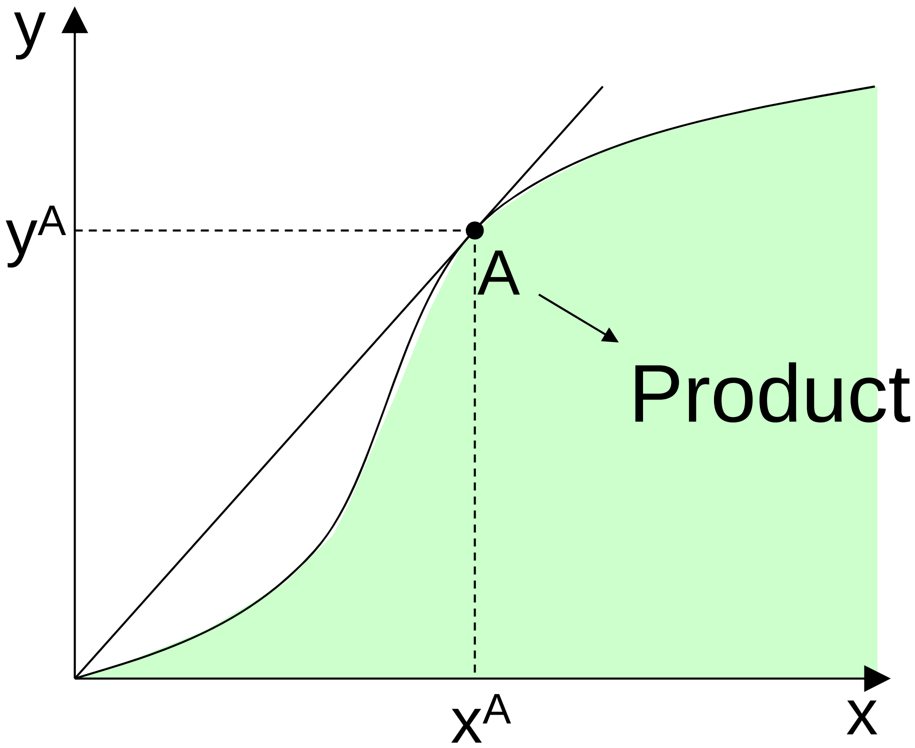
Productivity

$$\text{Productivity} = \frac{y}{x} = \frac{\textit{output}}{\textit{input}}$$



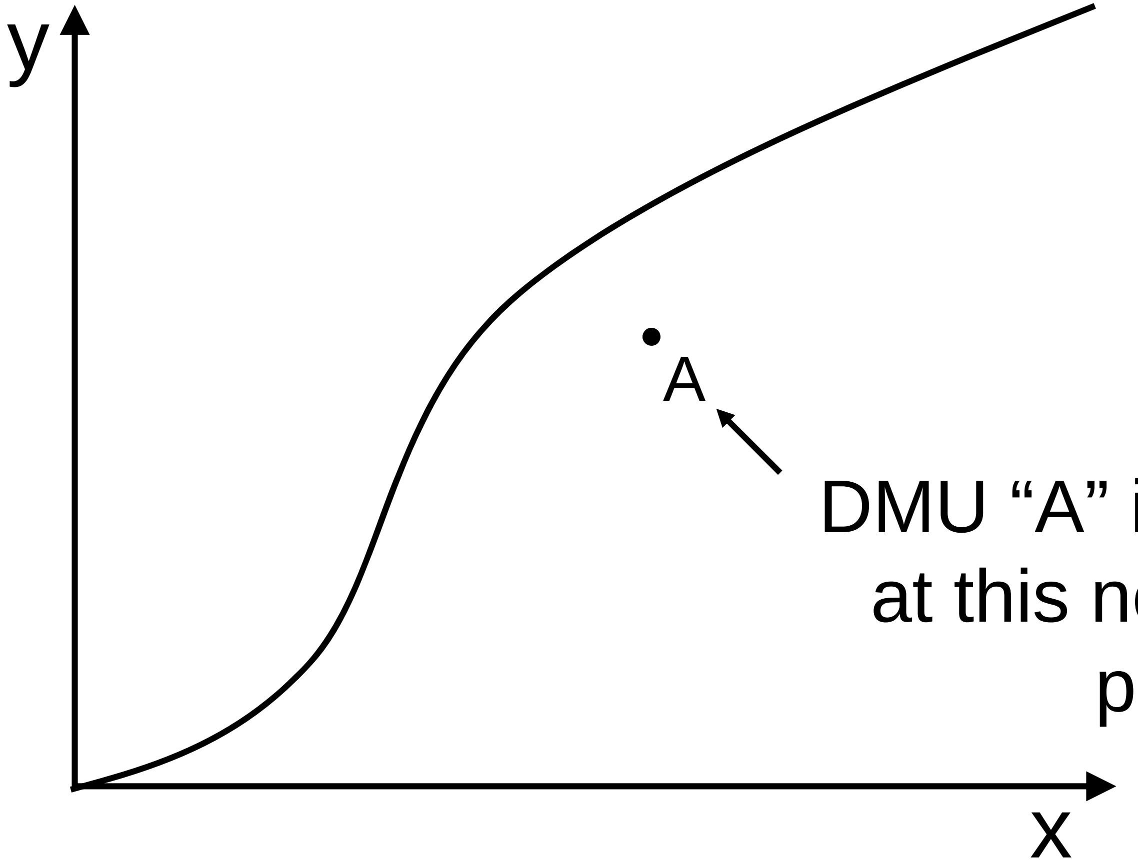
Productivity

$$\text{Productivity} = \frac{y}{x} = \frac{output}{input}$$



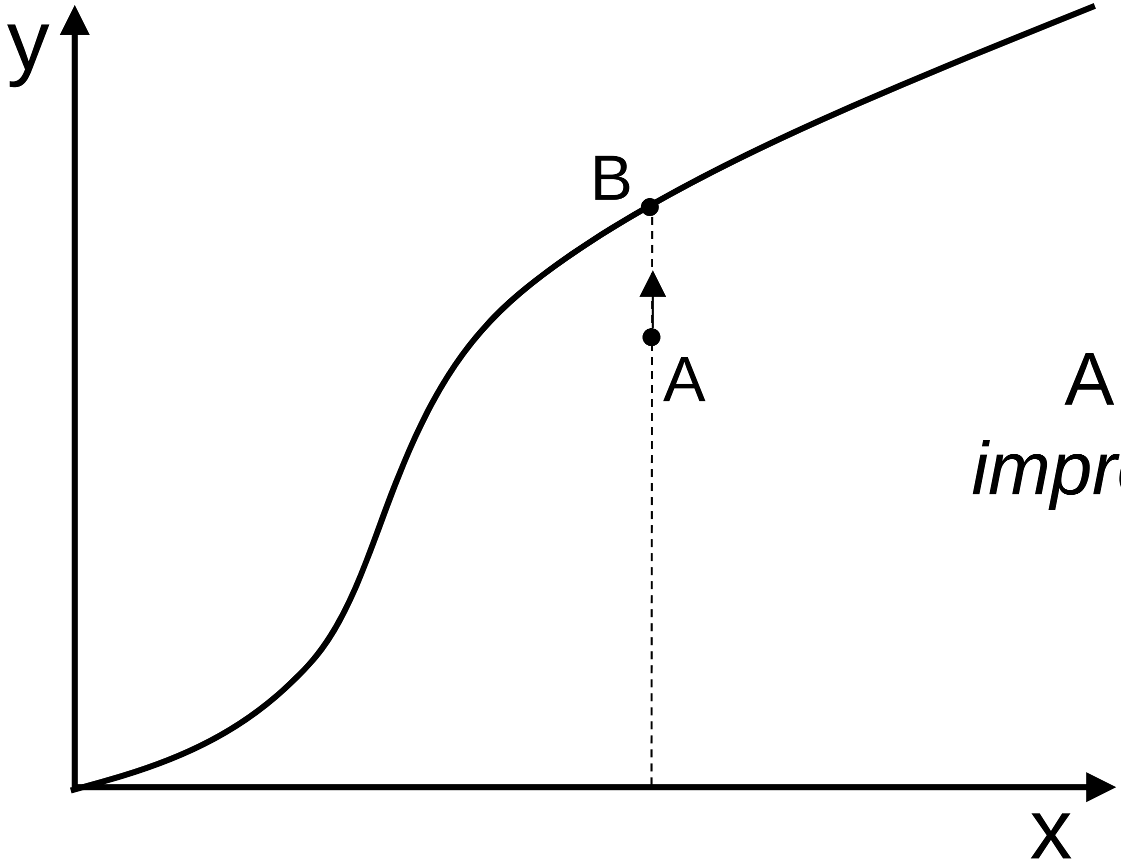
$$\text{Productivity A} = \frac{y^A}{x^A}$$

Productivity = Efficiency ?



DMU “A” is producing at this non-optimal point

Productivity = Efficiency ?

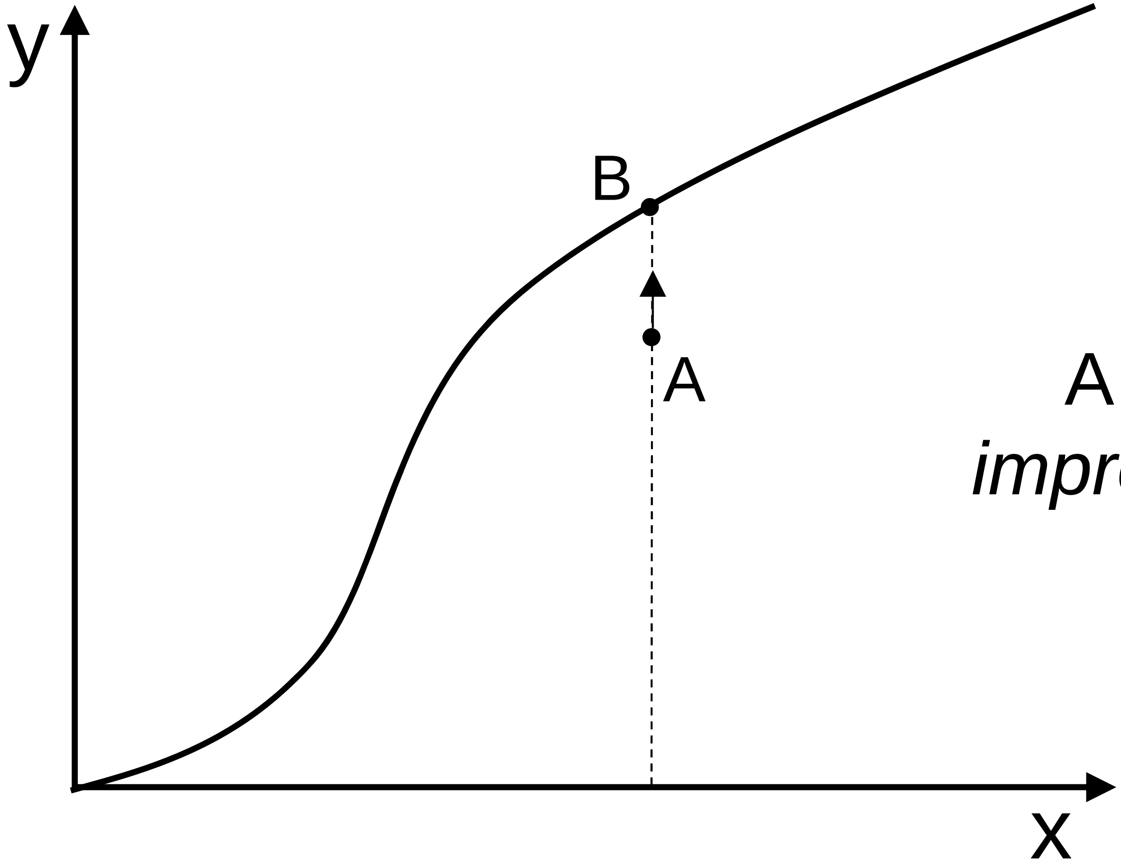


A → B means *improving technical efficiency*

Point B is *technically efficient* (= max output, given a certain input)

Moving from A to B is called *technical efficiency improvement.*

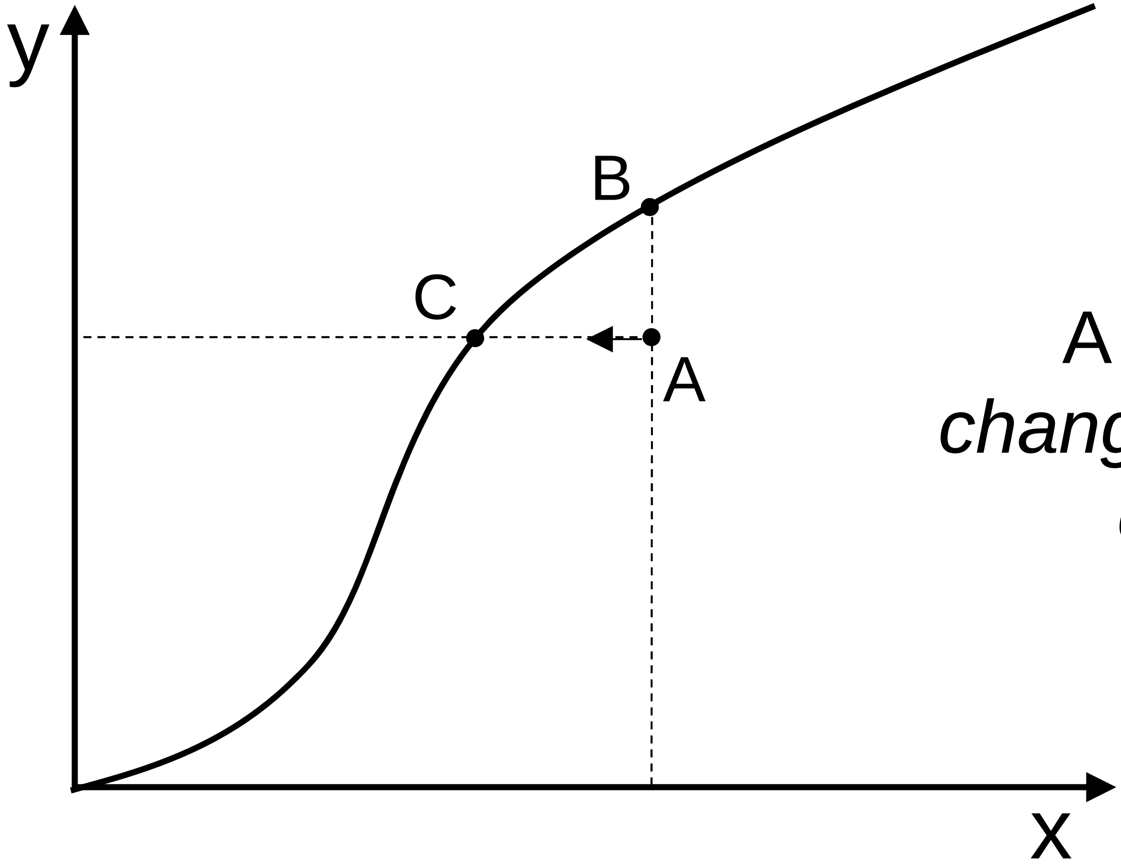
Productivity = Efficiency ?



A → B means
improving technical efficiency

Technical efficiency is to produce as much as possible with the technology currently present

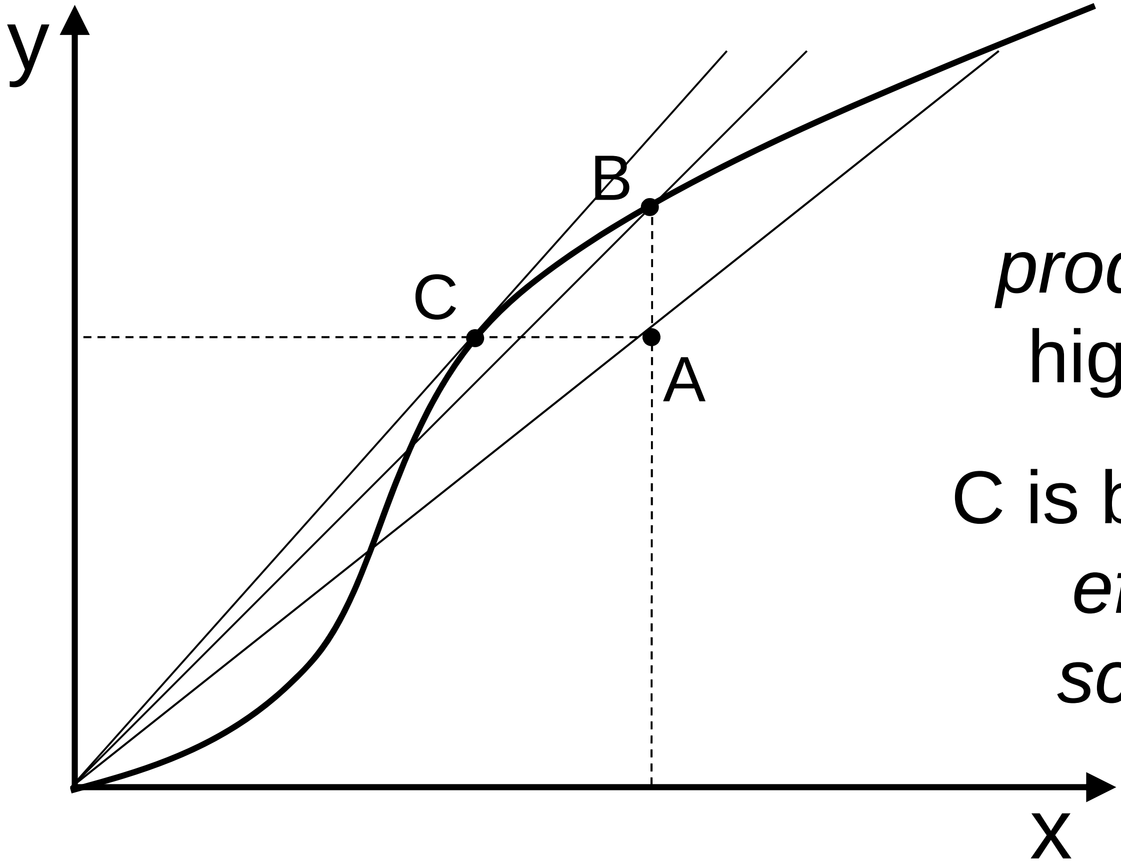
Productivity = Efficiency ?



A → C means *changing the scale of operations*

Moving from A to C is called *exploiting scale economies*

Productivity = Efficiency ?



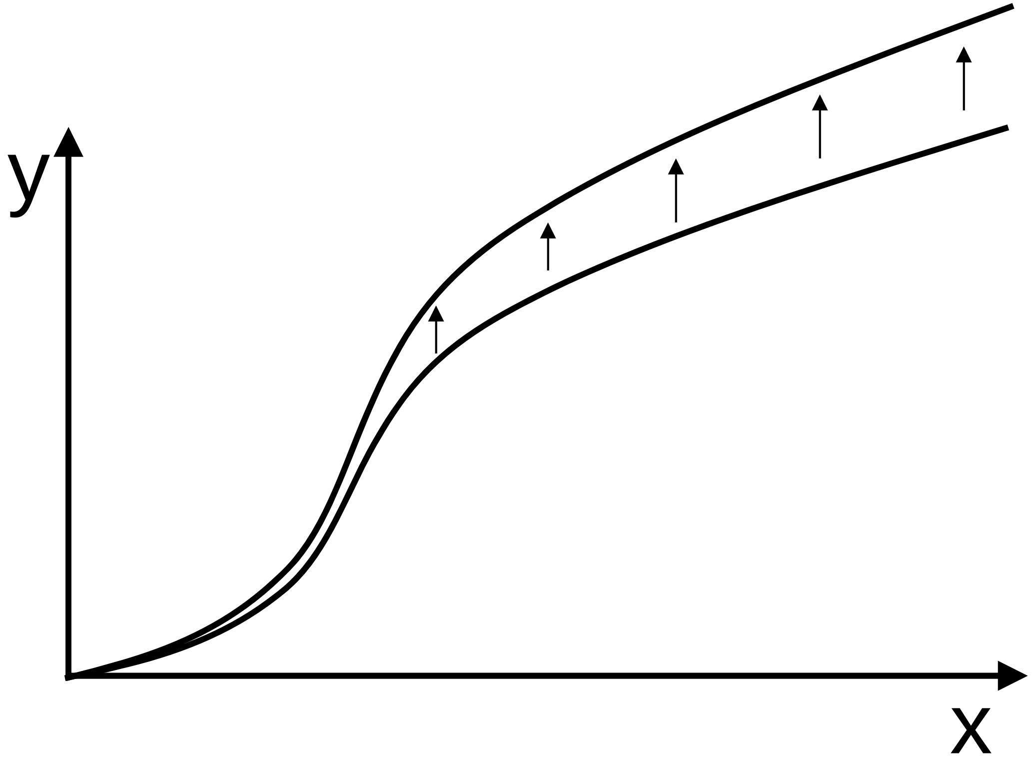
Note: the *productivity* in C is higher than in B!

C is both *technically efficient*, and *scale efficient*

So a *technically efficient* DMU (B) may still improve its *productivity* (B → C)!

3rd possibility to improve productivity: technical change

An advance in technology that shifts the production frontier upwards is called a *technical change*. This, however, usually costs time.



Conclusion

There are 3 options to increase productivity:

1. improve *technical efficiency* (more output, given the input)
2. exploit *scale economy* (the same output with less input)
3. *technical change* (shift upwards the production frontier through e.g. an advance in technology)

Output vs. input orientation

- **Output orientation**:

  By how much can output quantities be proportionally expanded without changing the input quantities used?

- **Input orientation**:

  By how much can input quantities be proportionally reduced without changing the output quantities produced?

Question: where do most hospitals focus on?

Technical versus allocative efficiency

Technical efficiency:

to obtain the maximum output, given a certain input

Allocative efficiency (involves prices or costs)*:*

to select a mix of inputs that produces a given output at *minimum cost*.

Economic efficiency = technical × allocative efficiency

Productivity

- Productivity is easy to define when you have 1 input and 1 output

- For multiple inputs and outputs, you need more ratios!

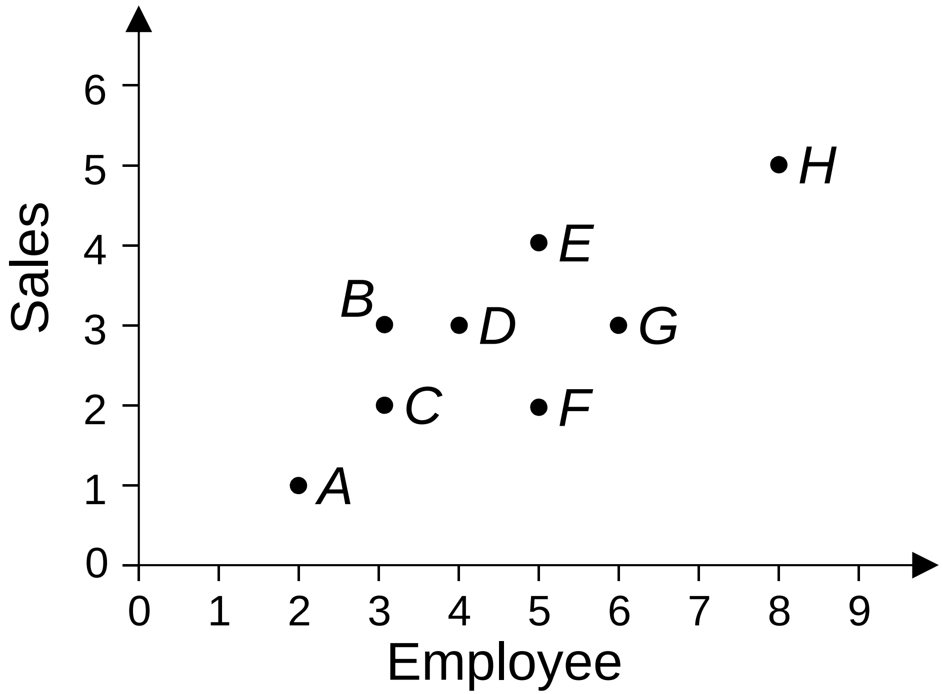
- Data Envelopment Analysis (DEA) is a technique to measure productivity with *multiple inputs and outputs*

Some definitions

- **DMU**: Decision Making Unit
  - E.g. hospital, firm, doctor, department
- **Input**: production means → *vector X*
  - E.g. labor, machine capacity, production costs
- **Output**: production → *vector Y*
  - E.g. treated patients, products, profit

Example: 1 input, 1 output

Store	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***
Employee (x)	2	3	3	4	5	5	6	8
Sale (y)	1	3	2	3	4	2	3	5



When is a store better than another?

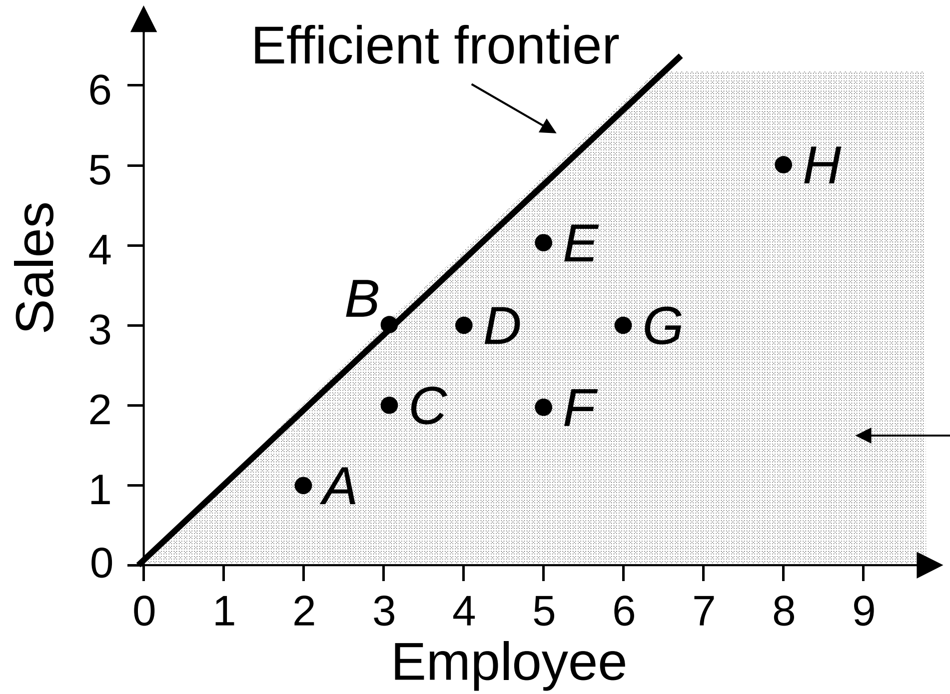
Which store performs the best?

Let's look at productivity ratios

Store	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***
Employee (x)	2	3	3	4	5	5	6	8
Sale (y)	1	3	2	3	4	2	3	5
Sale/Employee	0.5	1	0.667	0.75	0.8	0.4	0.5	0.625

Productivity

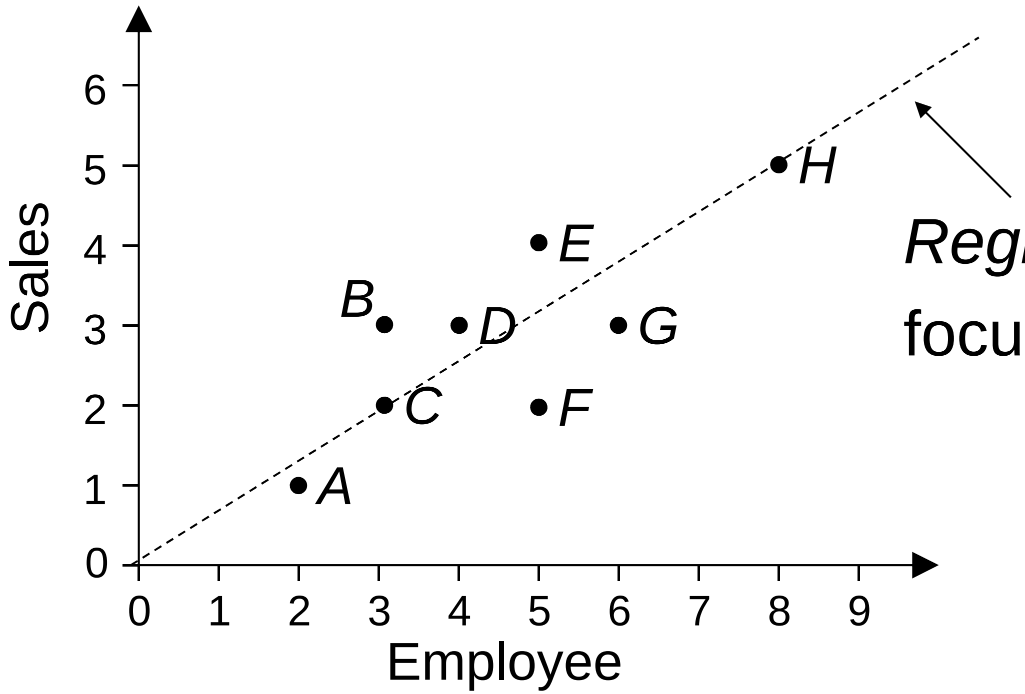
(€ / employee)



Envelopment area:
all feasible production combinations

Statistical regression line

Store	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***
Employee (x)	2	3	3	4	5	5	6	8
Sale (y)	1	3	2	3	4	2	3	5



Regression line (y = 0.622·x):

focuses on averages:

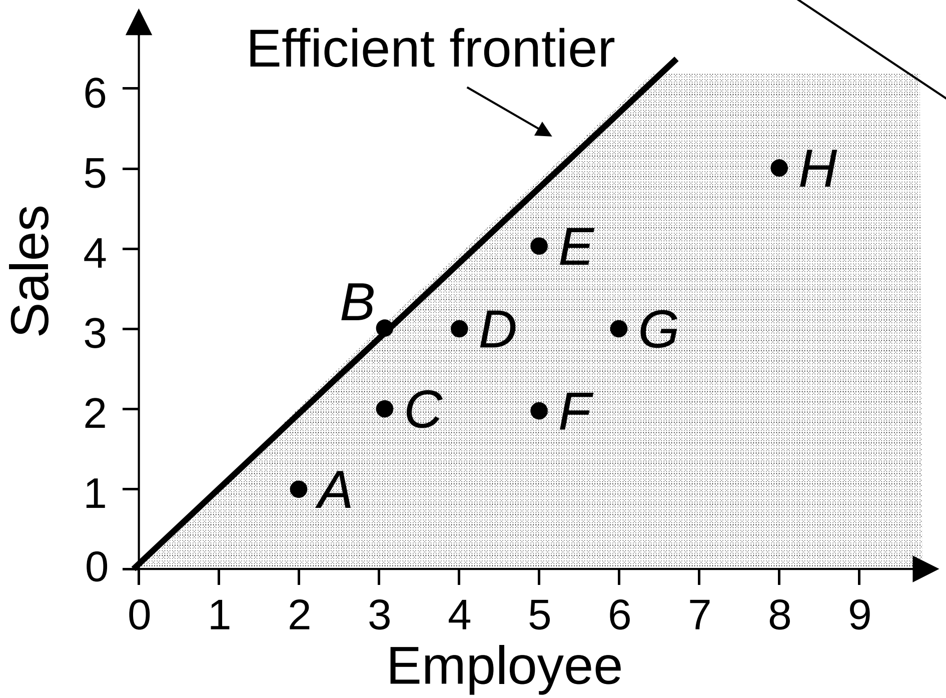
points above: excellent

points below: inferior

Data envelopment analysis

Store	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***
Employee (x)	2	3	3	4	5	5	6	8
Sale (y)	1	3	2	3	4	2	3	5
Sale/Employee	0.5	1	0.667	0.75	0.8	0.4	0.5	0.625

Productivity



In DEA, the best DMUs (***B***) serve as a benchmark

Efficiency others:

$$0 \leq \frac{\text{sale / employee}}{\text{sale B / employee B}} \leq 1$$

Data envelopment analysis

Store	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***
Employee (x)	2	3	3	4	5	5	6	8
Sale (y)	1	3	2	3	4	2	3	5
Sale/Employee	0.5	1	0.667	0.75	0.8	0.4	0.5	0.625

Efficiency others: $0 \leq \dfrac{\text{sale / employee}}{\text{sale B / employee B}} \leq 1$

For example: *efficiency F* is: 0.4/1 = 40%

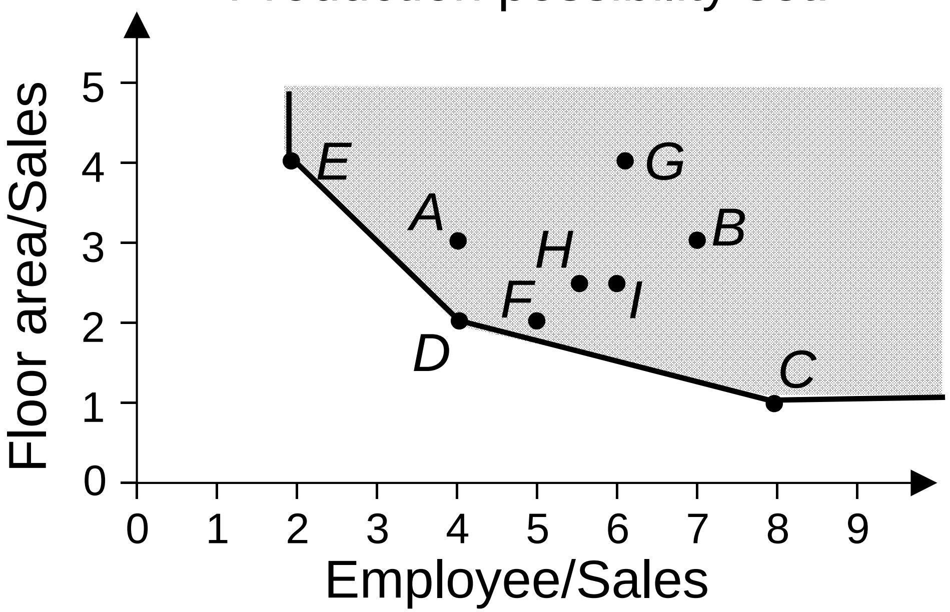
And: 1 = ***B*** > ***E*** > ***D*** > ***C*** > ***H*** > ***A*** = ***G*** > ***F*** = 0.4

This *relative efficiency measure* is *units invariant*, whereas *productivity ratio y/x* is **not** (*%* versus *€/employee*)!

Example: 2 inputs, 1 output

Store	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***	***I***
Employee	4	7	8	4	2	5	6	5.5	6
Floor area	3	3	1	2	4	2	4	2.5	2.5
Sale	1	1	1	1	1	1	1	1	1

Production possibility set:



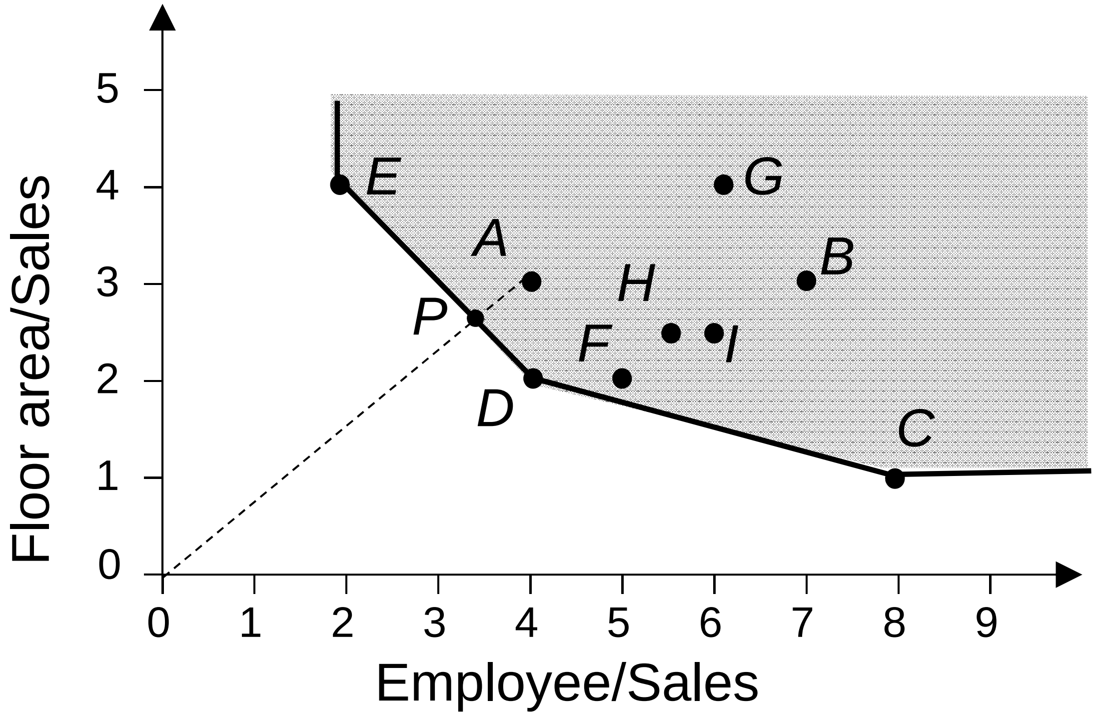
Efficient frontier: ***C-D-E***

C: the best w.r.t. floor area

E: the best w.r.t. employee

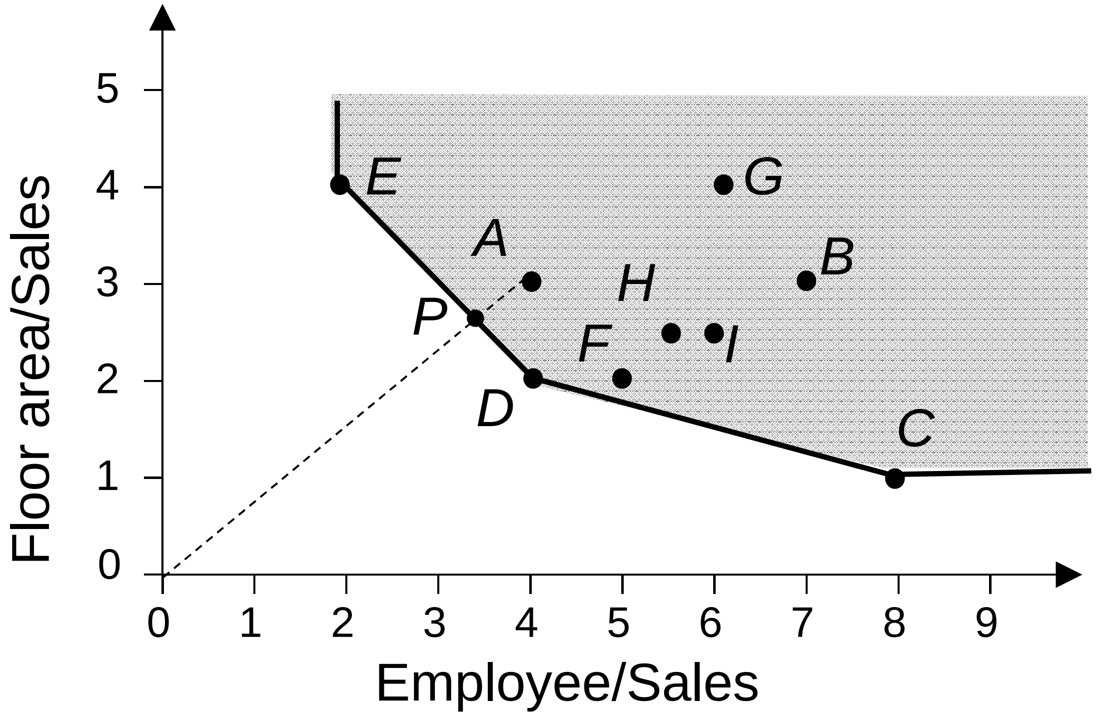
D: best combination

Example: 2 inputs, 1 output



- Efficiency of e.g. ***A*** is: $\dfrac{OP}{OA} = 0.8571$, or 85.71%
- ***A*** must multiply both inputs by 85.71% to become technically efficient (*input orientation*)

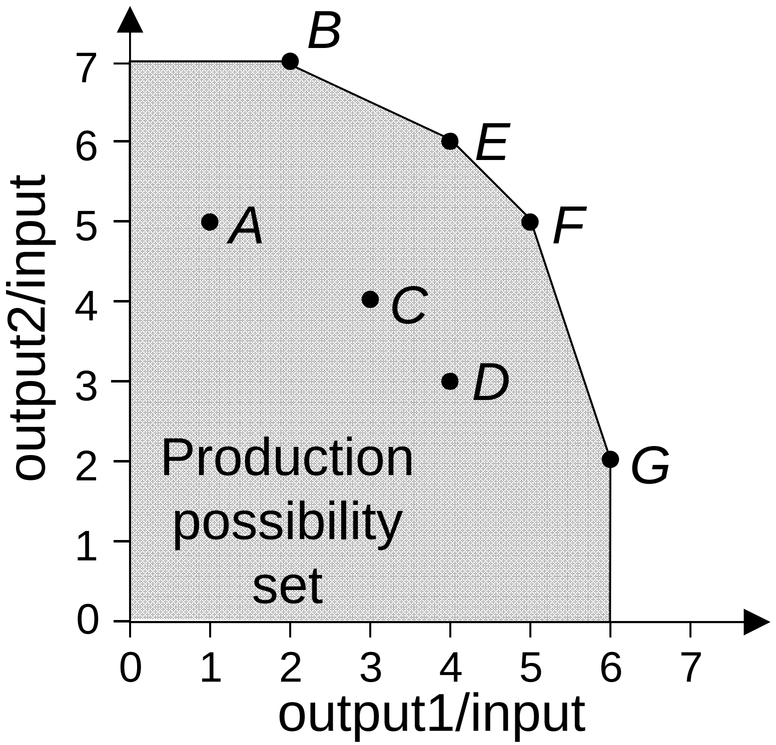
Example: 2 inputs, 1 output



- ***D*** and ***E*** are the *peers* / *reference set* of ***A***
  - ***A*** must consider ***D*** and ***E*** as examples to become technically efficient
- ***C*** and ***D*** are the *peers* / *reference set* of F, etc.

Example: 1 input, 2 outputs

Store	***A***	***B***	***C***	***D***	***E***	***F***	***G***
Employee	1	1	1	1	1	1	1
Customers	1	2	3	4	4	5	6
Sale	5	7	4	3	6	5	2



Efficient frontier: ***B*-*E*-*F*-*G***

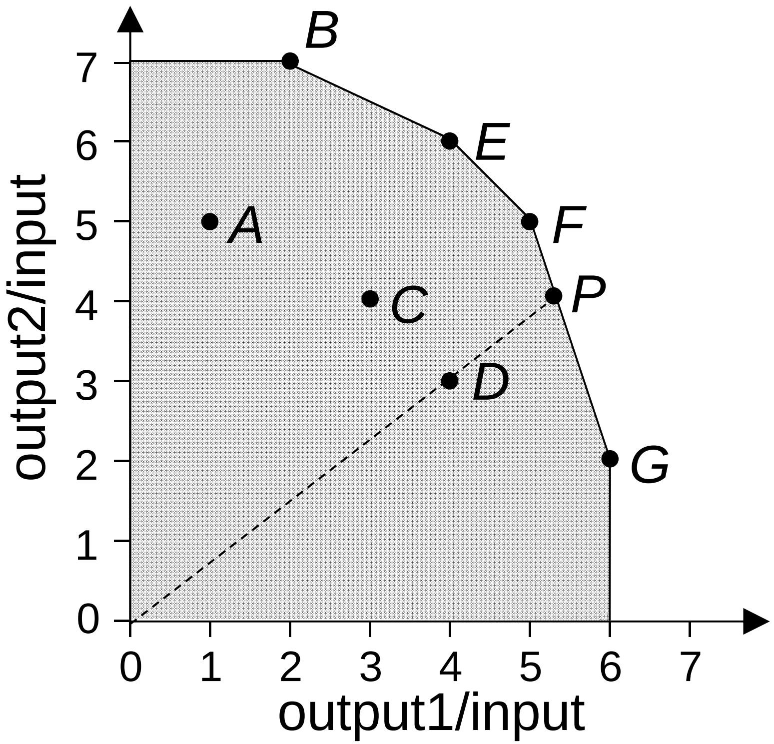
B: best w.r.t. sales

E: best combination

F: best combination

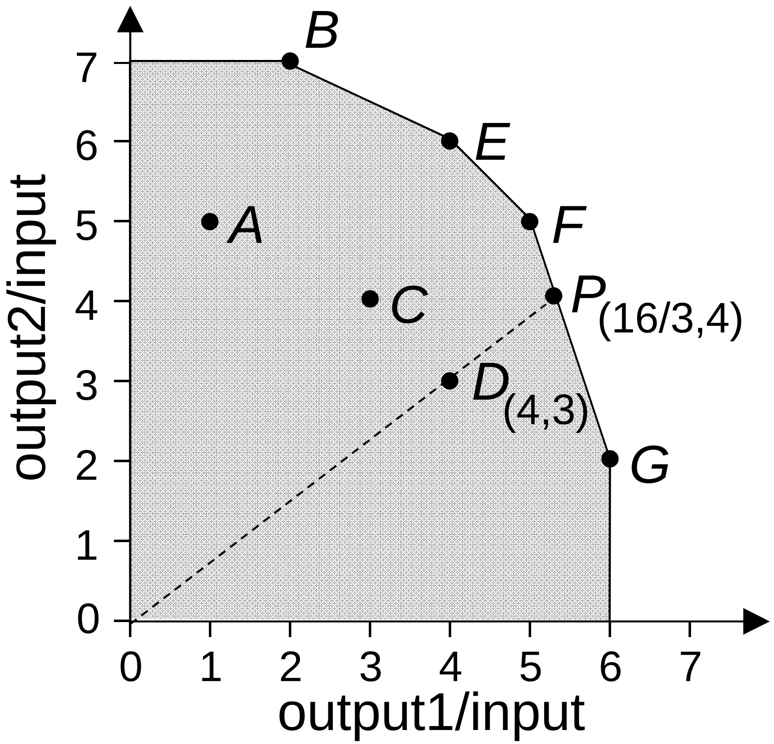
G: best w.r.t. customers

Example: 1 input, 2 outputs



- ***A***, ***C***, and ***D*** are inefficient
- Efficiency of ***D*** = ***OD*** / ***OP*** = 0.75 = 75%
- Reference set of ***D*** is {***F***, ***G***}

Example: 1 input, 2 outputs

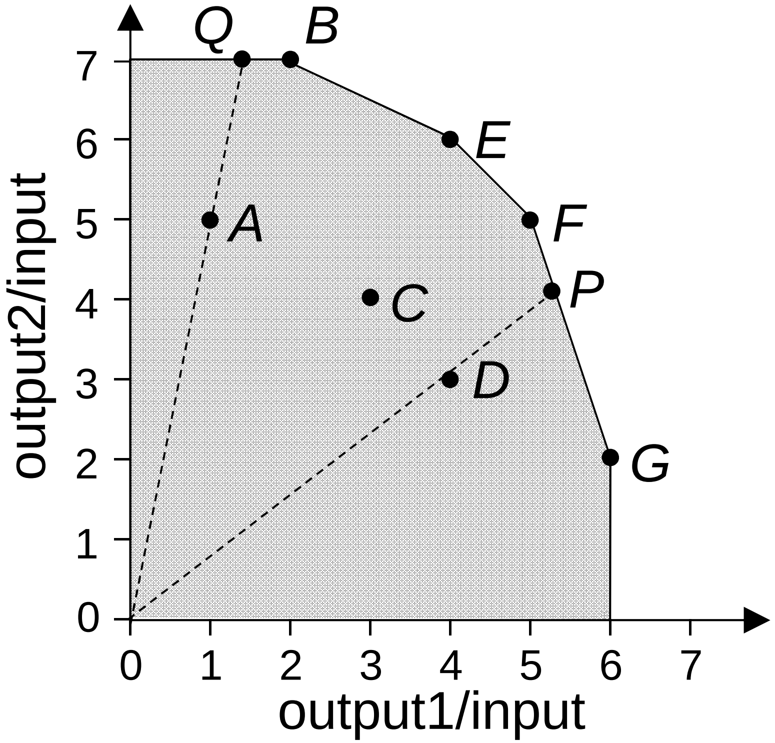


- Efficiency of ***D*** is 75%
- ***D*** must multiply its outputs by 1/0.75 = 1.33 to become technically efficient (*output orientation)*

  ➔1.33 x (4,3) = (16/3,4) = ***P***

Technical vs. mix inefficiency

- ***D* → *P*** eliminates inefficiency without changing the output/input proportions. ***D*** is called *technically inefficient*.
- ***A* → *Q*** shows that ***A*** is technically inefficient.



However, ***Q***'s output1 can be improved to ***B***, without modifying the input. This is called *mix inefficiency*.

➔ ***A*** is both *mix* and *technically inefficient*

Q is techn. efficient, but not mix efficient. ***B*** is both.

Example: 2 inputs, 2 outputs

Hospital	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***	***I***	***J***	***K***	***L***
Doctors	**20**	**19**	**25**	**27**	**22**	**55**	**33**	**31**	**30**	**50**	**53**	**38**
Nurses	**151**	**131**	**160**	**168**	**158**	**255**	**235**	**206**	**244**	**268**	**306**	**284**
Outpatients	**100**	**150**	**160**	**180**	**94**	**230**	**220**	**152**	**190**	**250**	**160**	**250**
Inpatients	**90**	**50**	**55**	**72**	**66**	**90**	**88**	**80**	**100**	**100**	**147**	**120**

When is a hospital better than another?

Which hospital performs the best?

Example: 2 inputs, 2 outputs

Hospital	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
Doctors	20	19	25	27	22	55	33	31	30	50	53	38
Nurses	151	131	160	168	158	255	235	206	244	268	306	284
Outpatients	100	150	160	180	94	230	220	152	190	250	160	250
Inpatients	90	50	55	72	66	90	88	80	100	100	147	120

Possibility: assign *weights* to inputs and outputs:

E.g. Weight for doctor : weight for nurse = **5 : 1**

Weight for outpatient : weight for inpatient = **1 : 3**

Drawback: these *fixed weights* are arbitrary

DEA: Data Envelopment Analysis
(Charnes, Cooper, Rhodes, 1978)

Hospital	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***	***I***	***J***	***K***	***L***
Doctors	**20**	**19**	**25**	**27**	**22**	**55**	**33**	**31**	**30**	**50**	**53**	**38**
Nurses	**151**	**131**	**160**	**168**	**158**	**255**	**235**	**206**	**244**	**268**	**306**	**284**
Outpatients	**100**	**150**	**160**	**180**	**94**	**230**	**220**	**152**	**190**	**250**	**160**	**250**
Inpatients	**90**	**50**	**55**	**72**	**66**	**90**	**88**	**80**	**100**	**100**	**147**	**120**

- *DEA* uses *variable weights*
- Each hospital chooses weights individually, such that:
  - Their productivity is maximized, and between **0** and **1**
  - All other DMU productivities are also between **0** and **1**

DEA notation

Entities:

N DMUs (index i, j)

K inputs

M outputs

Parameters:

$\mathbf{x_i}$ = K×1 input vector for DMU i

$\mathbf{y_i}$ = M×1 output vector for DMU i

Variables:

$\mathbf{v}$ = K×1 input weight vector

$\mathbf{u}$ = M×1 output weight vector

DEA notation: example

Hospital	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
Doctors	**20**	**19**	**25**	**27**	**22**	**55**	**33**	**31**	**30**	**50**	**53**	**38**
Nurses	**151**	**131**	**160**	**168**	**158**	**255**	**235**	**206**	**244**	**268**	**306**	**284**
Inpatients	**90**	**50**	**55**	**72**	**66**	**90**	**88**	**80**	**100**	**100**	**147**	**120**

K = 2 inputs, M = 1 output

$\mathbf{x}_\mathbf{i}$ = K×1 = 2×1 input vector for DMU *i*

$\mathbf{y}_\mathbf{i}$ = M×1 = 1×1 output vector for DMU *i*

Constant Returns to Scale (CRS) DEA model

CRS: linear relation between input and output

For each DMU *i*, we want to obtain a vector **v** of input weights and a vector **u** of output weights, such that the weighed ratio of outputs and inputs is maximal:

$$\max \frac{u^T y_i}{v^T x_i} \quad \begin{matrix} \longleftarrow \text{Weighed output} \\ \longleftarrow \text{Weighed input} \end{matrix}$$

Where **u**, **v** are variable vectors, and $\mathbf{x_i}$, $\mathbf{y_i}$ are resp. input and output parameter vectors

CRS-DEA model

Basic form (model solved for each DMU j):

$$\boxed{\begin{aligned} &\max \frac{u^T y_i}{v^T x_i} \\ &\text{s.t.} \quad \frac{u^T y_i}{v^T x_i} \leq 1 \left(\forall i\right) \\ &\qquad u, v \geq 0 \end{aligned}}$$

In other words: find values for the weight vectors **u** and **v**, in such a way that the efficiency measure for this DMU j is maximized

CRS-DEA model

If (**u***,**v***) is a solution, then (α·**u***, α·**v***) is also a solution $\Rightarrow$ there are infinitely many solutions!

Solution: add constraints: $\mathbf{v^T x_i = 1}$, and rewrite the model as follows:

Ratio form of DEA model

$$\max \frac{u^T y_i}{v^T x_i}$$

$$\text{s.t.} \quad \frac{u^T y_i}{v^T x_i} \leq 1 \left(\forall i\right)$$

$$u, v \geq 0$$

$\rightarrow$

Multiplier form of DEA model

$$\max \mu^T y_i$$

s.t.

$$\nu^T x_i = 1$$

$$\mu^T y_i - \nu^T x_i \leq 0 \left(\forall i\right)$$

$$\mu, \nu \geq 0$$

CRS-DEA model

v and **u** are replaced by ν and μ respectively, to discern between these two models

Ratio form of DEA model

$$\max \frac{u^T y_i}{v^T x_i}$$

$$\text{s.t.} \quad \frac{u^T y_i}{v^T x_i} \leq 1 \left(\forall i\right)$$

$$u, v \geq 0$$

$\rightarrow$

Multiplier form of DEA model

$$\max \mu^T y_i$$

s.t.

$$\nu^T x_i = 1$$

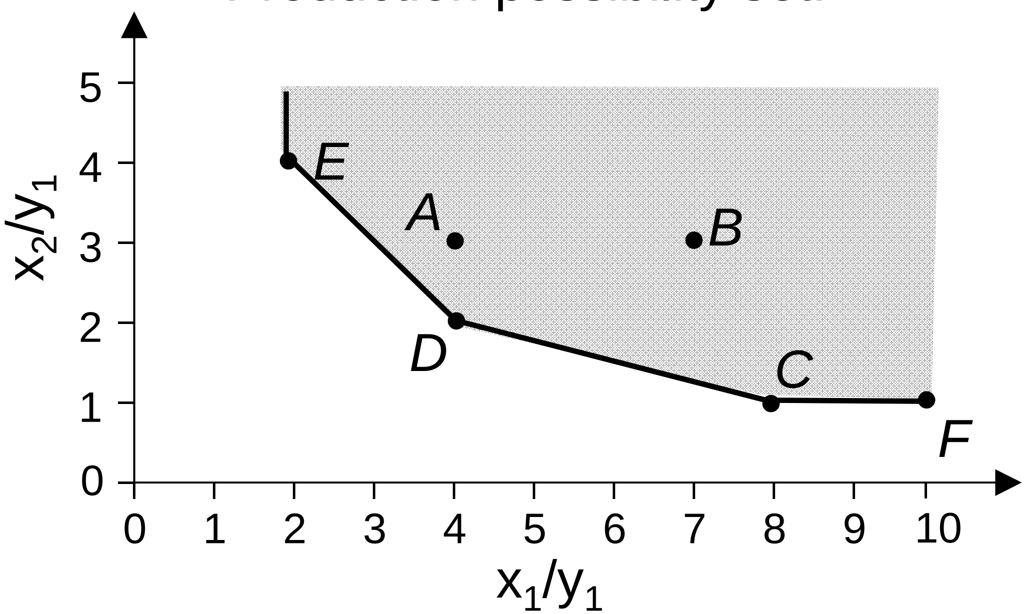
$$\mu^T y_i - \nu^T x_i \leq 0 \left(\forall i\right)$$

$$\mu, \nu \geq 0$$

Example: 2 inputs, 1 output

	DMU	A	B	C	D	E	F
Input:	x_1	4	7	8	4	2	10
	x_2	3	3	1	2	4	1
Output:	y_1	1	1	1	1	1	1

Production possibility set:



Example: 2 inputs, 1 output

	DMU	A	B	C	D	E	F
Input:	x_1	4	7	8	4	2	10
	x_2	3	3	1	2	4	1
Output:	y_1	1	1	1	1	1	1

$$
\begin{aligned}
&\max \mu^T y_i \\
&\text{s.t.} \quad \nu^T x_i = 1 \\
&\mu^T y_i - \nu^T x_i \le 0 \ (\forall i) \\
&\mu, \nu \ge 0
\end{aligned}
$$

for DMU B:

$$\max \mu \cdot 1$$

$$7\nu_1 + 3\nu_2 = 1$$

$$\mu \le 4\nu_1 + 3\nu_2 \ (A)$$

$$\mu \le 7\nu_1 + 3\nu_2 \ (B)$$

$$\mu \le 8\nu_1 + \nu_2 \ (C)$$

$$\mu \le 4\nu_1 + 2\nu_2 \ (D)$$

$$\mu \le 2\nu_1 + 4\nu_2 \ (E)$$

$$\mu \le 10\nu_1 + \nu_2 \ (F)$$

$$\mu, \nu \ge 0$$

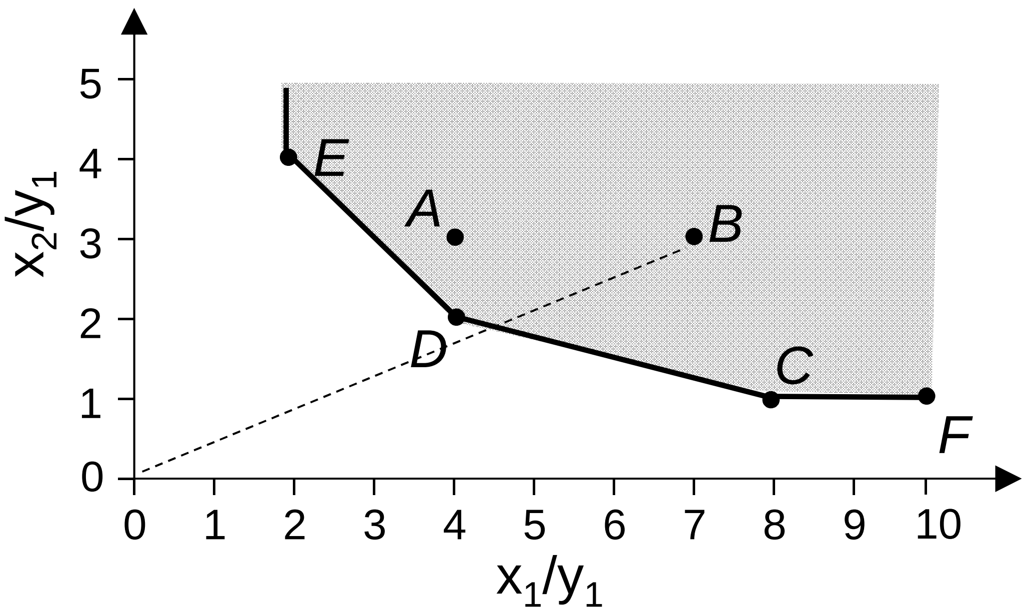
Example: 2 inputs, 1 output

Optimal solution for B:

$\nu_1 = 0.0526$, $\nu_2 = 0.2105$, $\mu = 0.6316 = \theta^* =$ efficiency

Reference set: {***C***, ***D***} (equality occurs for constraints (***C***) and (***D***))

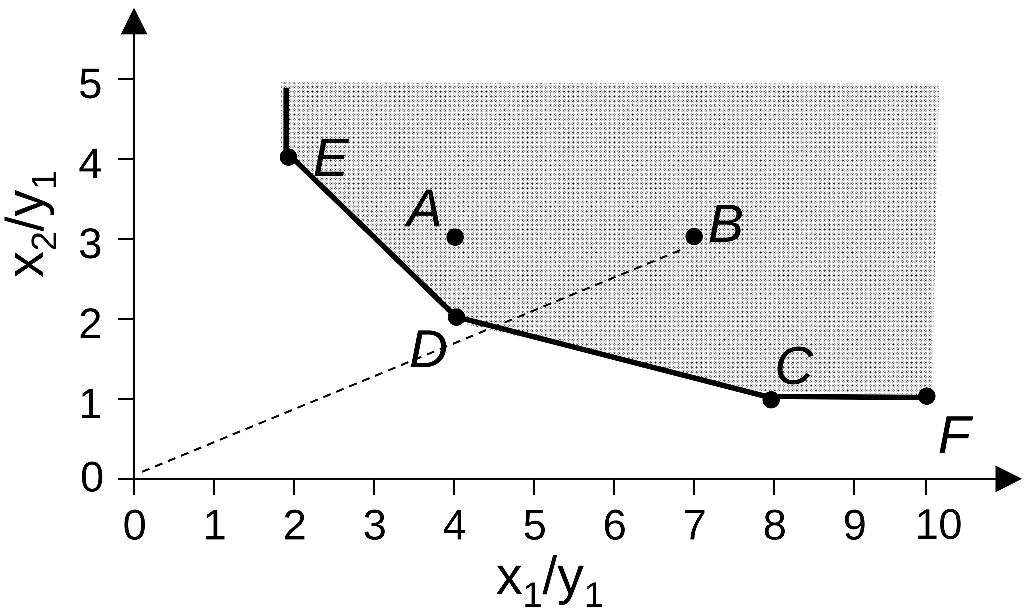
Note: $\nu_2 / \nu_1 = 4$ → input2 is four times as important to ***B*** as input1



Example: 2 inputs, 1 output

*Optimal solution for **F**:*

$\nu_1 = 0$, $\nu_2 = 1$, $\mu = 1 = \theta^* =$ efficiency



Pareto-Koopmans efficiency

Definition:

A DMU is fully efficient if and only if it is not possible to improve any input or output without worsening some other input or output

CRS-DEA model: envelopment form

Dual form of multiplier model is the envelopment form:

(input oriented version)

Multiplier form of DEA model → Envelopment form of DEA model

$$\begin{aligned} &\max \mu^T y_i \\ &\text{s.t.} \\ &\nu^T x_i = 1 \\ &\mu^T y_i - \nu^T x_i \le 0 \ (\forall i) \\ &\mu, \nu \ge 0 \end{aligned}$$

1 dual: θ

N duals: λ

$$\begin{aligned} &\min \theta \\ &\text{s.t.} \\ &-y_i + Y\lambda \ge 0 \\ &\theta x_i - X\lambda \ge 0 \\ &\theta \text{ free}, \lambda \ge 0 \end{aligned}$$

1 + N constraints > K + M constraints

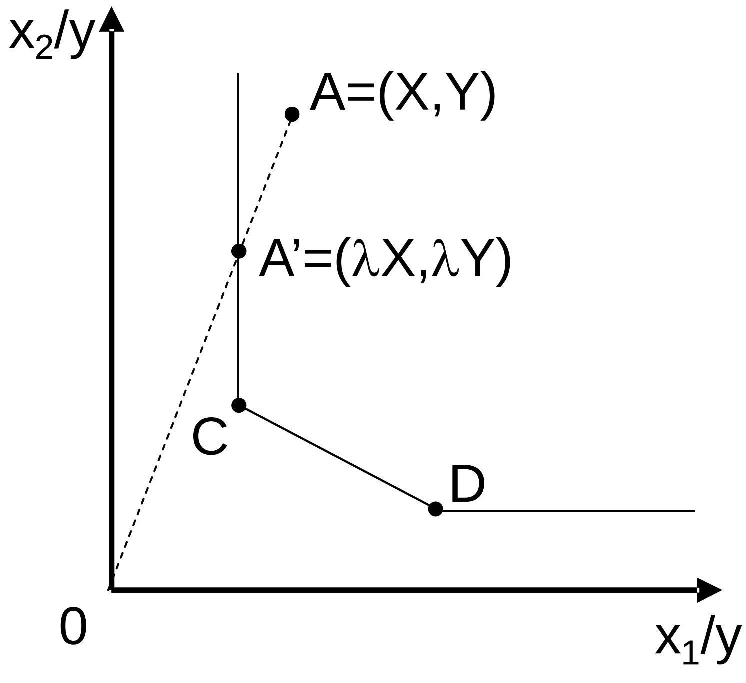
easier!

Envelopment form: interpretation

θ = efficiency score for DMU j (θ ≤ 0)

The model radially contracts input vector x_i, while remaining in the feasible input area.

$$
\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&-y_i + Y\lambda \geq 0 \\
&\theta x_i - X\lambda \geq 0 \\
&\theta \text{ free}, \lambda \geq 0
\end{aligned}
$$



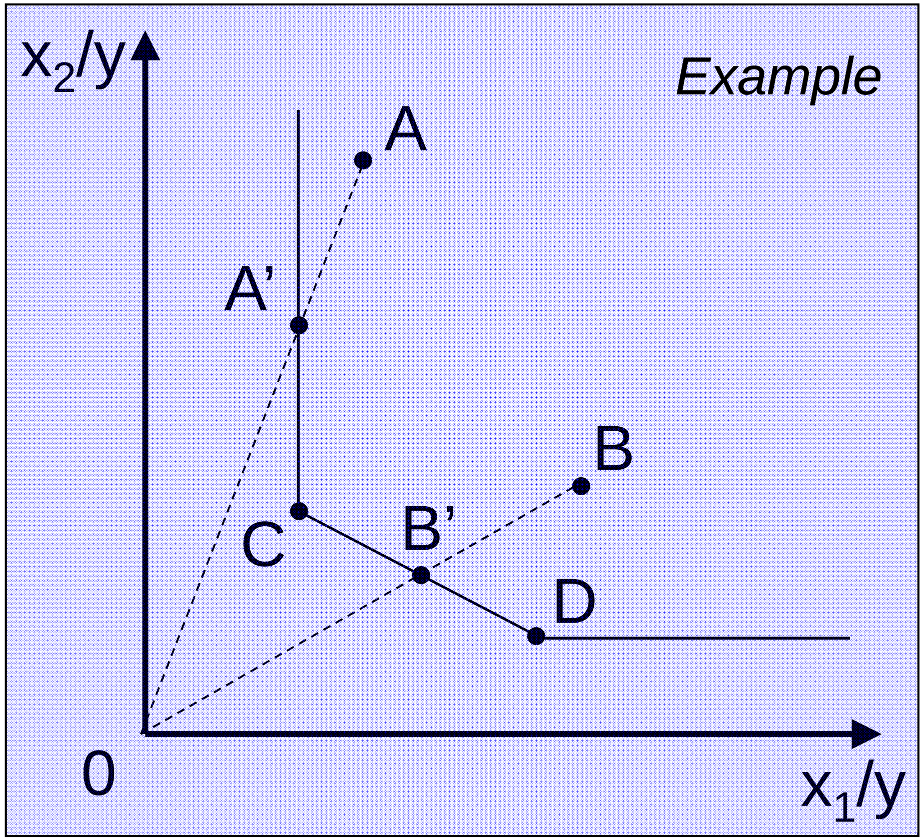
The radial contraction of input vector x_i produces a projected point (Xλ,Yλ) on the efficient frontier.

Projection and *slacks*

The projection of the inefficient DMUs is:

A to A’

B to B’



However, it is questionable whether A’ is an efficient point, since its *input* x_2 can be reduced without changing the output (A’ → C).

This is known as *input slack*. Similarly, there can also be *output slack.* Slack = mix inefficiency!

Projection and *slacks*

Envelopment form

$$\boxed{\begin{array}{l} \min \theta \\ \text{s.t.} \\ -y_i + Y\lambda \geq 0 \\ \theta x_i - X\lambda \geq 0 \\ \theta \text{ free}, \lambda \geq 0 \end{array}}$$

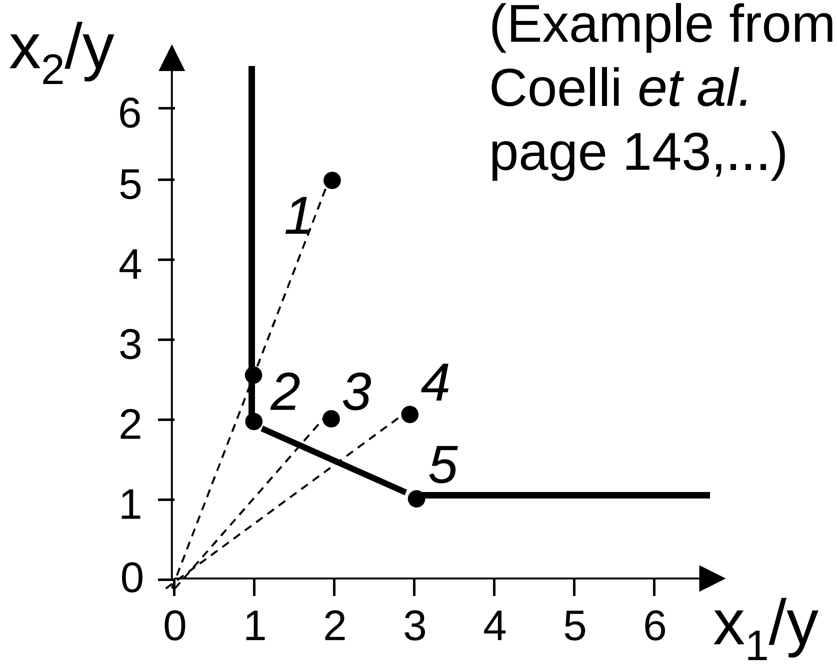
$-y_i + Y\lambda \geq 0 \rightarrow -y_i + Y\lambda =$ Output slack

$\theta x_i - X\lambda \geq 0 \rightarrow \theta x_i - X\lambda =$ Input slack

Example: 2 inputs, 1 output

	DMU	1	2	3	4	5
Input:	x_1	2	2	6	3	6
	x_2	5	4	6	2	2
Output:	y	1	2	3	1	2
	x_1/y	2	1	2	3	3
	x_2/y	5	2	2	2	1

(Example from Coelli *et al.* page 143,...)



$$\min \theta$$
s.t.
$$-y_i + Y\lambda \geq 0$$
$$\theta x_i - X\lambda \geq 0$$
$$\theta \text{ free}, \lambda \geq 0$$

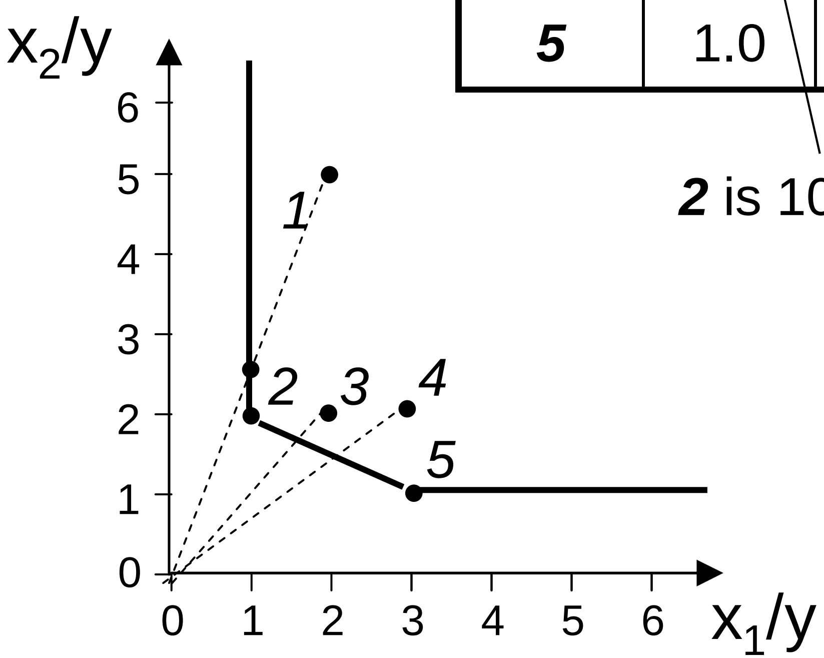
For DMU ***3***:

$$\min \theta$$
$$\text{s.t.} \quad -y_3 + (y_1\lambda_1 + y_2\lambda_2 + \ldots + y_5\lambda_5) \geq 0$$
$$\theta x_{13} - (x_{11}\lambda_1 + x_{12}\lambda_2 + \ldots + x_{15}\lambda_5) \geq 0$$
$$\theta x_{23} - (x_{21}\lambda_1 + x_{22}\lambda_2 + \ldots + x_{25}\lambda_5) \geq 0$$
$$\theta \text{ free}, (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \geq 0$$

Example: 2 inputs, 1 output

DEA results:

DMU	θ	λ_1	λ_2	λ_3	λ_4	λ_5	IS_1	IS_2	OS
1	0.5	-	0.5	-	-	-	-	0.5	-
2	1.0	-	1.0	-	-	-	-	-	-
3	0.833	-	1.0	-	-	0.5	-	-	-
4	0.714	-	0.214	-	-	0.286	-	-	-
5	1.0	-	-	-	-	1.0	-	-	-



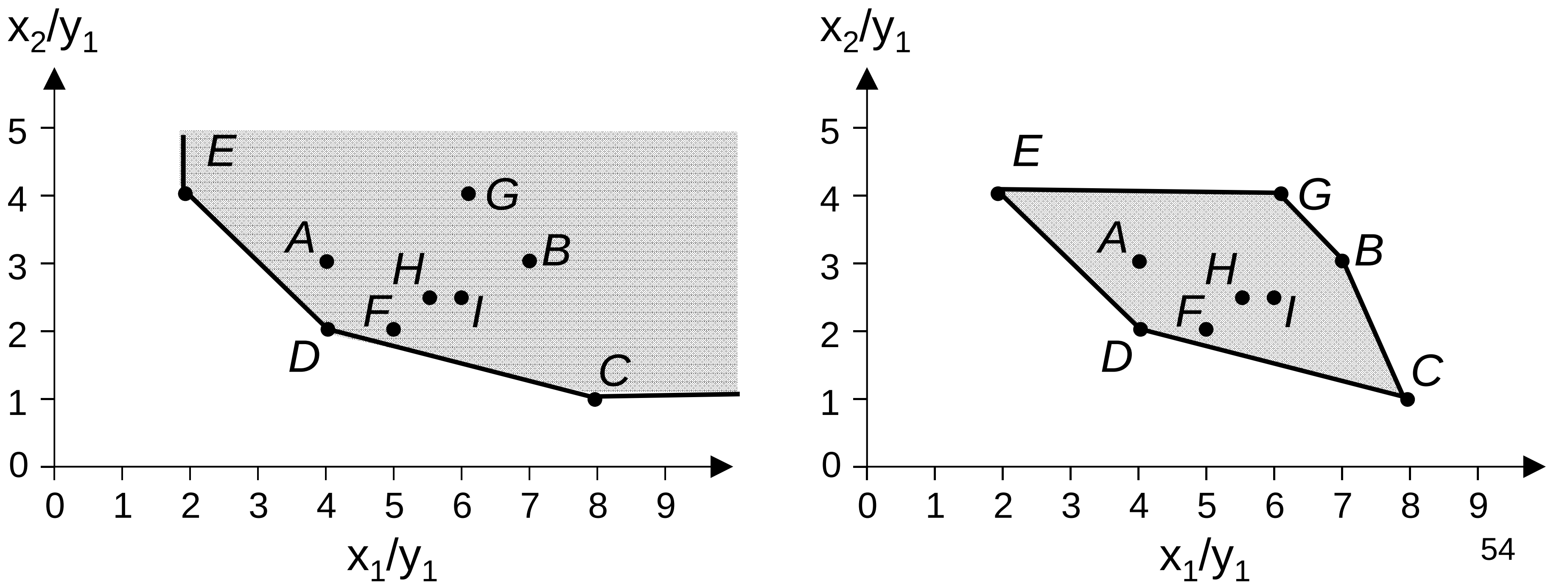
2 is 100% efficient

Reference set / targets / peers for ***3*** are {***2***, ***5***}

1 is 50% efficient, and has input slack (= technical and mix inefficiency)

Variable Returns to Scale (VRS)

- CRS: k times input means k times output. This may not be realistic (e.g. constraints on finance)!
- VRS: DEA only considers the *convex hull* of observed DMUs (DMUs are only benchmarked against DMUs of similar size)



Variable Returns to Scale (VRS)

CRS DEA model

$$\min \theta$$

s.t.

$$-y_i + Y\lambda \geq 0$$

$$\theta x_i - X\lambda \geq 0$$

$$\theta \text{ free}, \lambda \geq 0$$

VRS DEA model

$$\min \theta$$

s.t.

$$-y_i + Y\lambda \geq 0$$

$$\theta x_i - X\lambda \geq 0$$

$$1^T \lambda = 1$$

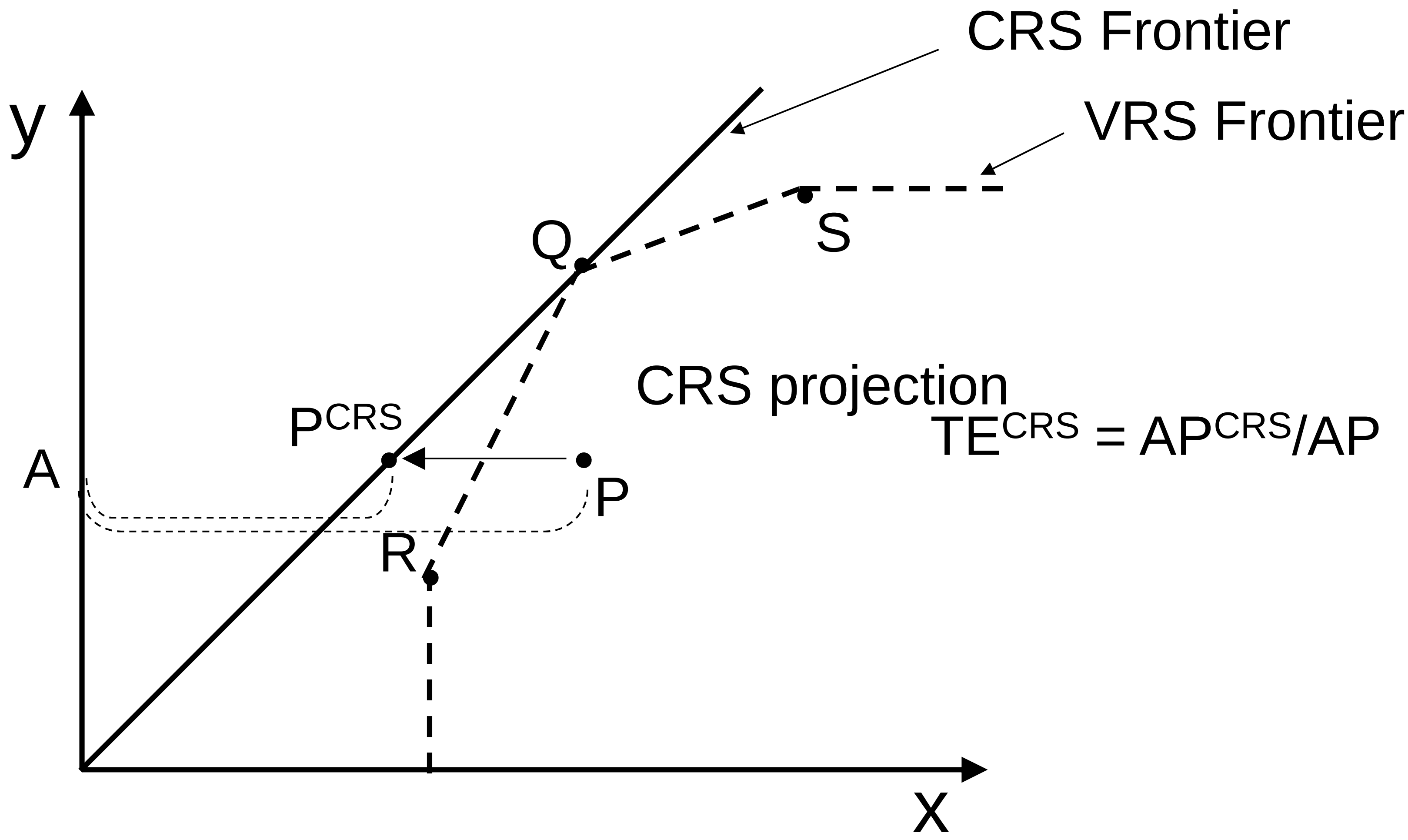
$$\theta \text{ free}, \lambda \geq 0$$

$$\Leftrightarrow \sum_i \lambda_i = 1$$

Convex combination of inputs and outputs

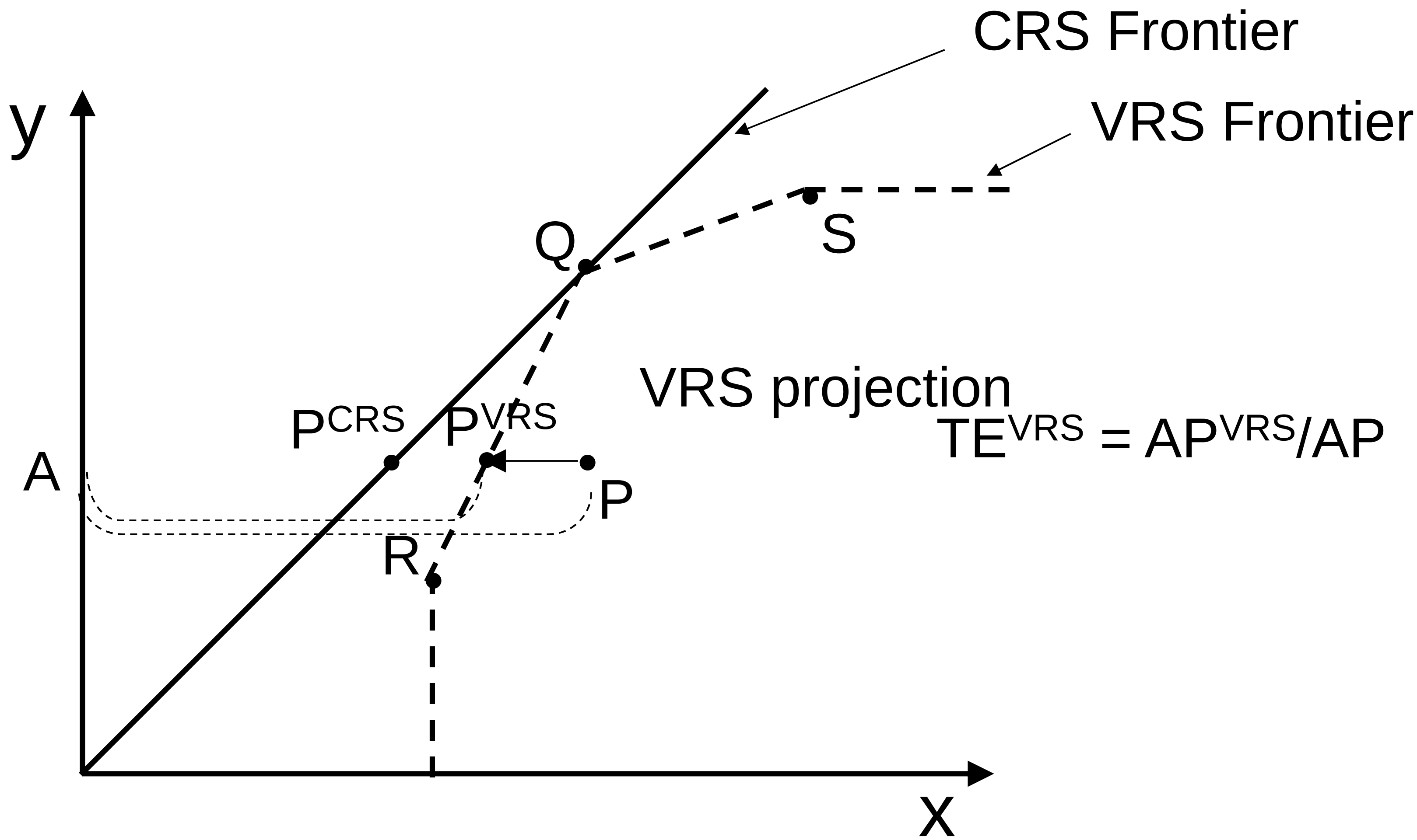
Scale efficiency

Example (1 input, 1 output):



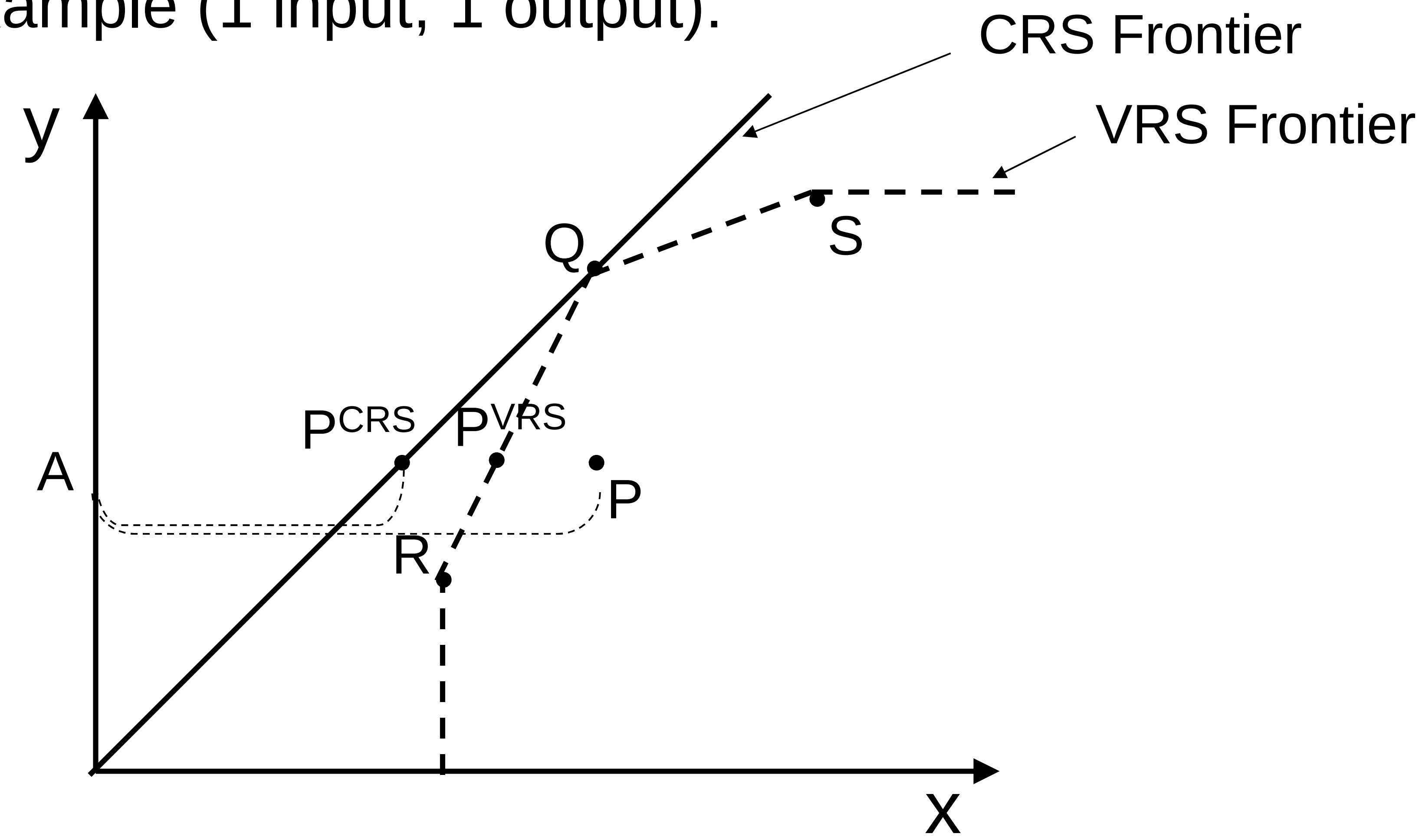
Scale efficiency

Example (1 input, 1 output):



Scale efficiency

Example (1 input, 1 output):



Scale efficiency (SE): AP^{CRS} / AP^{VRS} = TE^{CRS}/TE^{VRS}

Scale efficiency

This means that: $TE^{CRS} = TE^{VRS} \times SE$

In other words, CRS technical efficiency can be decomposed in *"pure" technical efficiency*, and *scale efficiency*.

Furthermore: the scale efficiency can be computed from the CRS and VRS solutions

Scale efficiency (SE): $AP^{CRS} / AP^{VRS} = TE^{CRS}/TE^{VRS}$

Input vs. output oriented DEA

Input oriented VRS DEA model		Output oriented VRS DEA model	
min θ		max ϕ	→ $1/\phi$ = TE score
s.t.		s.t.	
$-y_i + Y\lambda \geq 0$		$-\phi y_i + Y\lambda \geq 0$	
$\theta x_i - X\lambda \geq 0$	$\leftrightarrow$	$x_i - X\lambda \geq 0$	
$1^T \lambda = 1$		$1^T \lambda = 1$	
θ free, $\lambda \geq 0$		ϕ free, $\lambda \geq 0$	

$\boldsymbol{\phi}$ = proportional increase in output that can be achieved with inputs held constant

Extensions & variants of the model

- Output oriented model
- Non-discretionary inputs or outputs
  - Cannot be influenced by management
- Undesirable measure models
  - Outputs you want to minimize
    for example: cancellations, overtime
- Inclusion of prices: revenue max., or cost min. →
- Additive model
  - Combined input and output orientation
  - Add-efficient *iif* input and output slacks are 0
- Models with restricted multipliers

Revenue maximization

Output oriented VRS model	Revenue maximization model
$\max \phi$	$\max\ p_i y_i^*$
s.t.	s.t.
$-\phi y_i + Y\lambda \geq 0$	$-y_i^* + Y\lambda \geq 0$
$x_i - X\lambda \geq 0$	$x_i - X\lambda \geq 0$
$1^T \lambda = 1$	$1^T \lambda = 1$
ϕ free, $\lambda \geq 0$	y_i^* free, $\lambda \geq 0$

Economic efficiency: $\dfrac{p_i y_i}{p_i y_i^*}$

$p_i y_i$ ← Observed output

$p_i y_i^*$ ← Maximized output

Cost minimization

Input oriented VRS model

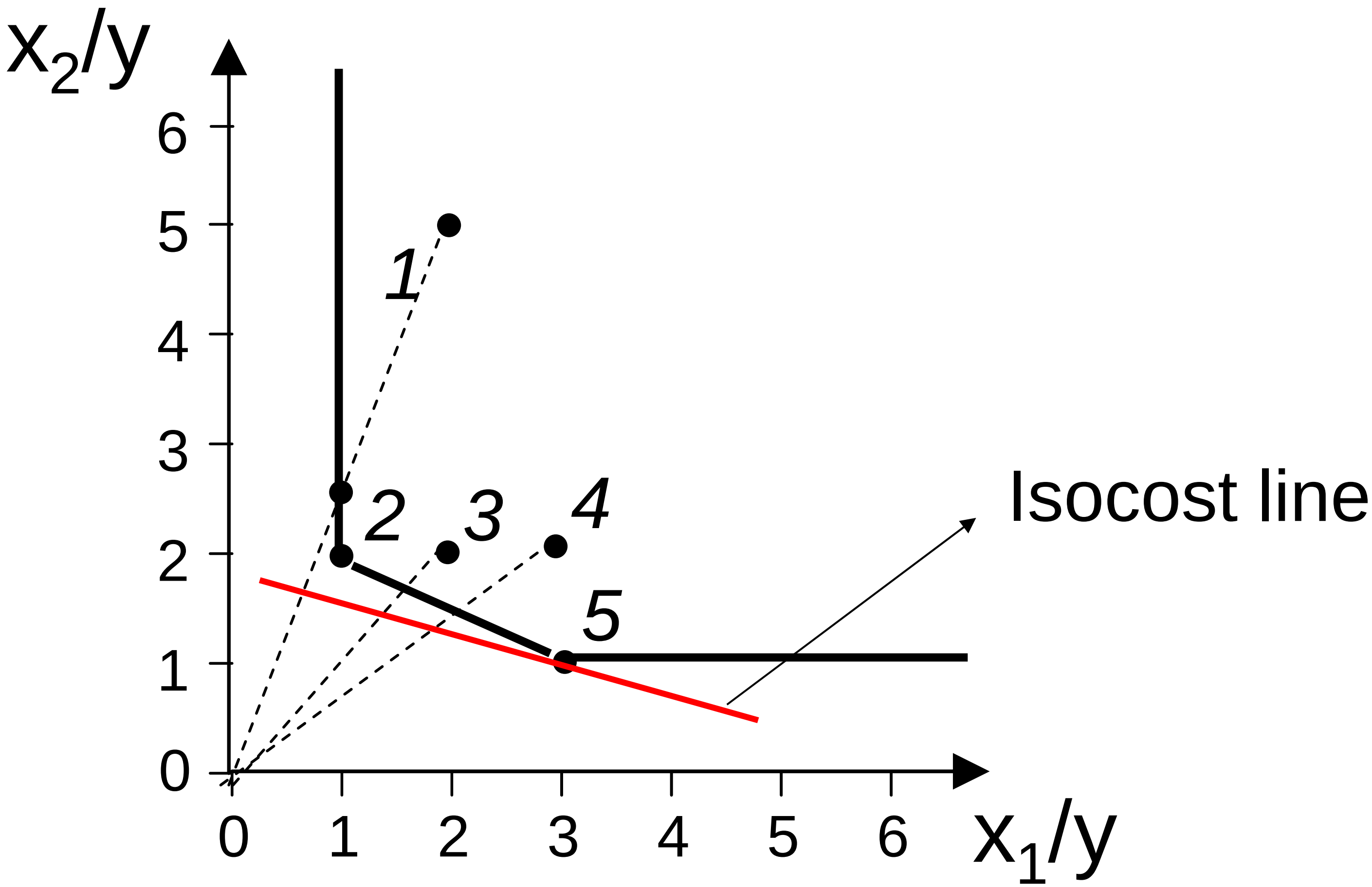
$$\begin{aligned} &\min \theta \\ &\text{s.t.} \\ &-y_i + Y\lambda \geq 0 \\ &\theta x_i - X\lambda \geq 0 \\ &\theta \text{ free}, \lambda \geq 0 \end{aligned}$$

$\longrightarrow$

Cost minimization model

$$\begin{aligned} &\min \ p_i x_i^* \\ &\text{s.t.} \\ &-y_i + Y\lambda \geq 0 \\ &x_i^* - X\lambda \geq 0 \\ &1^T \lambda = 1 \\ &x_i^* \text{ free}, \lambda \geq 0 \end{aligned}$$

Cost minimization example



Extensions & variants of the model

- Output oriented model
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  – Combined input and output orientation
  – Add-efficient *iif* input and output slacks are 0
- Models with restricted multipliers →

Models with restricted multipliers

- For inefficient DMUs, we may see zeros as optimal weights for inputs or outputs

  → may not be realistic
- Two approaches to deal with this:
  - *Assurance region method*
  - *Cone-ratio method*

Models with restricted multipliers
Assurance region method

- Add a constraint on the ratio of input weights:

$$lb \leq \frac{v_2}{v_1} \leq ub \quad \Leftrightarrow \quad v_1 \cdot lb \leq v_2 \wedge v_2 \leq v_1 \cdot ub$$

Definition

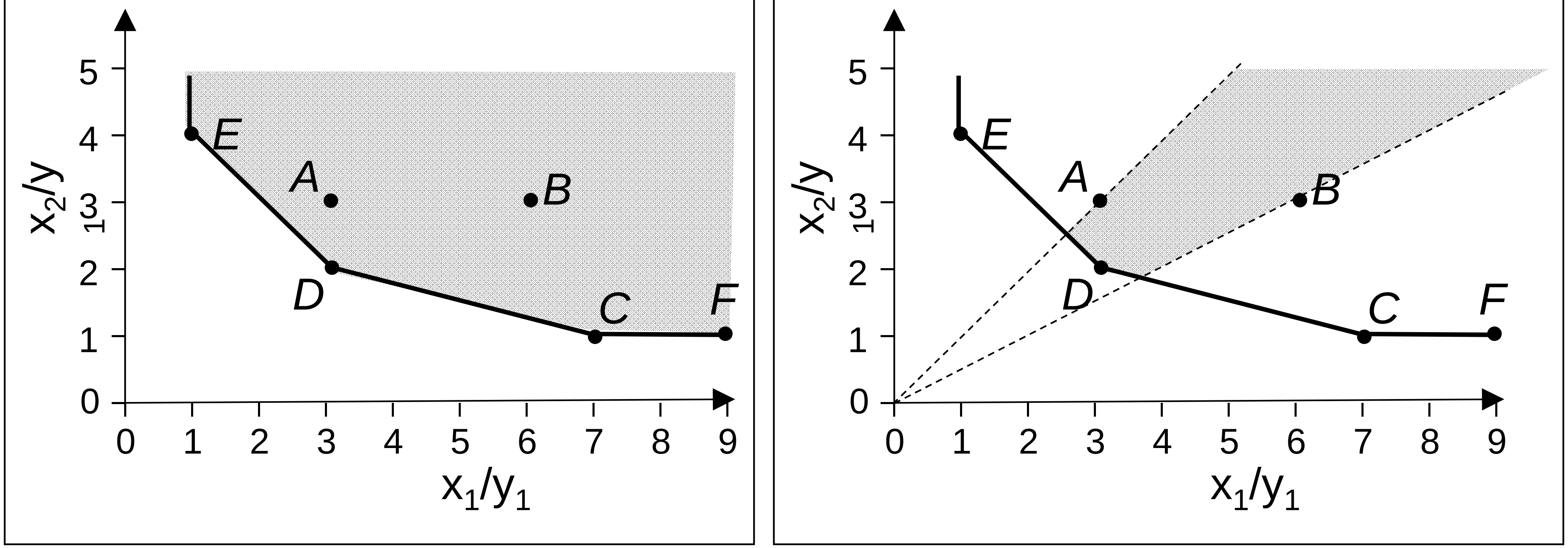
a DMU is *AR-efficient* if θ*=1 and all slacks are 0

Models with restricted multipliers

Assurance region method

$$0.5 \le \frac{\nu_2}{\nu_1} \le 1 \quad \Leftrightarrow$$

$$\nu_1 \cdot 0.5 \le \nu_2 \wedge \nu_2 \le \nu_1$$



Models with restricted multipliers
Cone-ratio method

- Generalization of the assurance region method
- Feasible region of the input vector **v** is restricted to a polyhedral convex cone, spanned by K vectors

$$v = \sum_{j=1}^{K} \alpha_j \cdot \overline{a_j} \quad (\alpha_j \geq 0, \forall j)$$

- Similarly, the output vector is restricted to another polyhedral convex cone

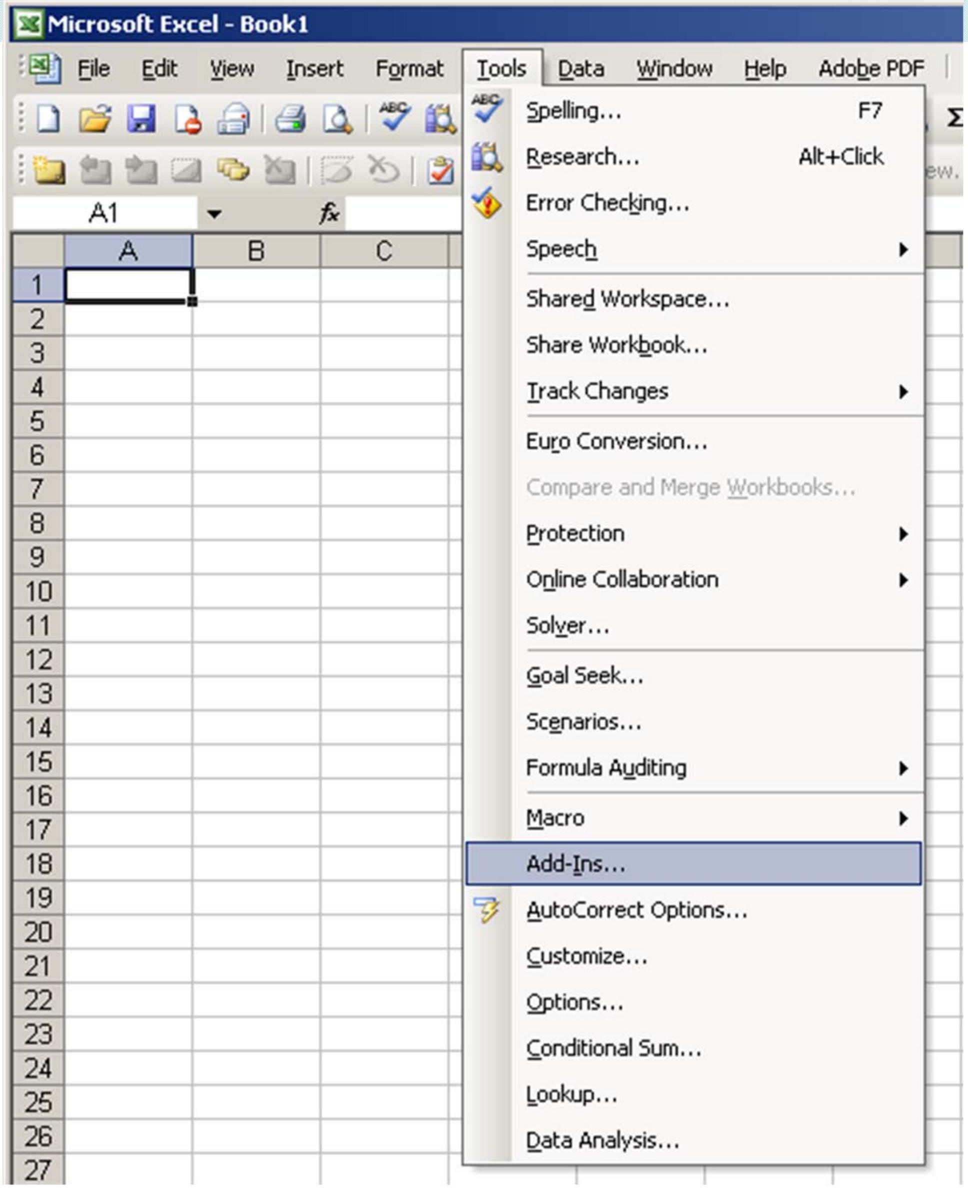
DEAFrontier

DEAFrontier TM consists of a series of DEA software which are Add-Ins for Microsoft® Excel, developed by Joe Zhu

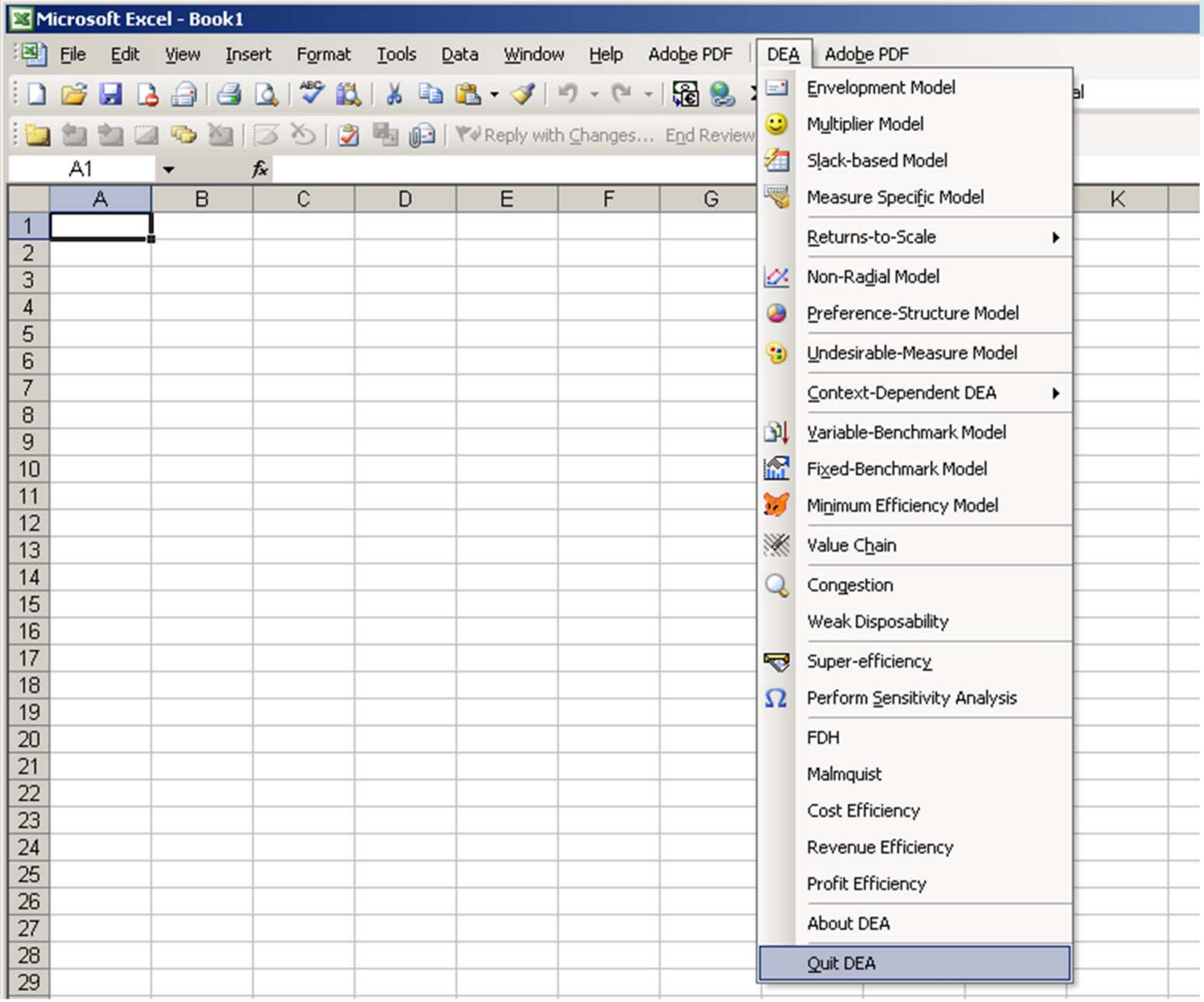
DEAFrontier uses Excel Solver, and requires Excel 97 or later versions and Windows 95 or higher.

See: http://www.deafrontier.com/software.html

Installation of DEAFrontier



Installation of DEAFrontier



Considerations

- Input orientation or output orientation?
- Undesirable measures?
- CRS model or VRS model?
- Restricted multipliers?
- Try many different models

  For example:

  – one that analyzes quality of care,

  – One that analyzes productivity

  → Use two efficiency scores in new DEA analysis

Considerations

- Many DMUs can end up with 100% efficiency
  - Reduce the number of inputs and/or outputs
    - Rule of thumb: every input or output requires 3 DMUs
    - Avoid input and/or output correlation
  - Add a virtual “super-DMU” which takes the best scores for each input and output
  - Do a “peer-count” for all efficient DMUs

- DEA is sensitive to data errors

Stochastic Frontier Analysis

- Based on Cobb-Douglas production frontier:

Vector with outputs

$$\boxed{\ln y_i = x_i^{'} \cdot \beta - e_i \quad (i = 1, \ldots, N)}$$

Vector with logarithms of inputs

Vector of unknown parameters

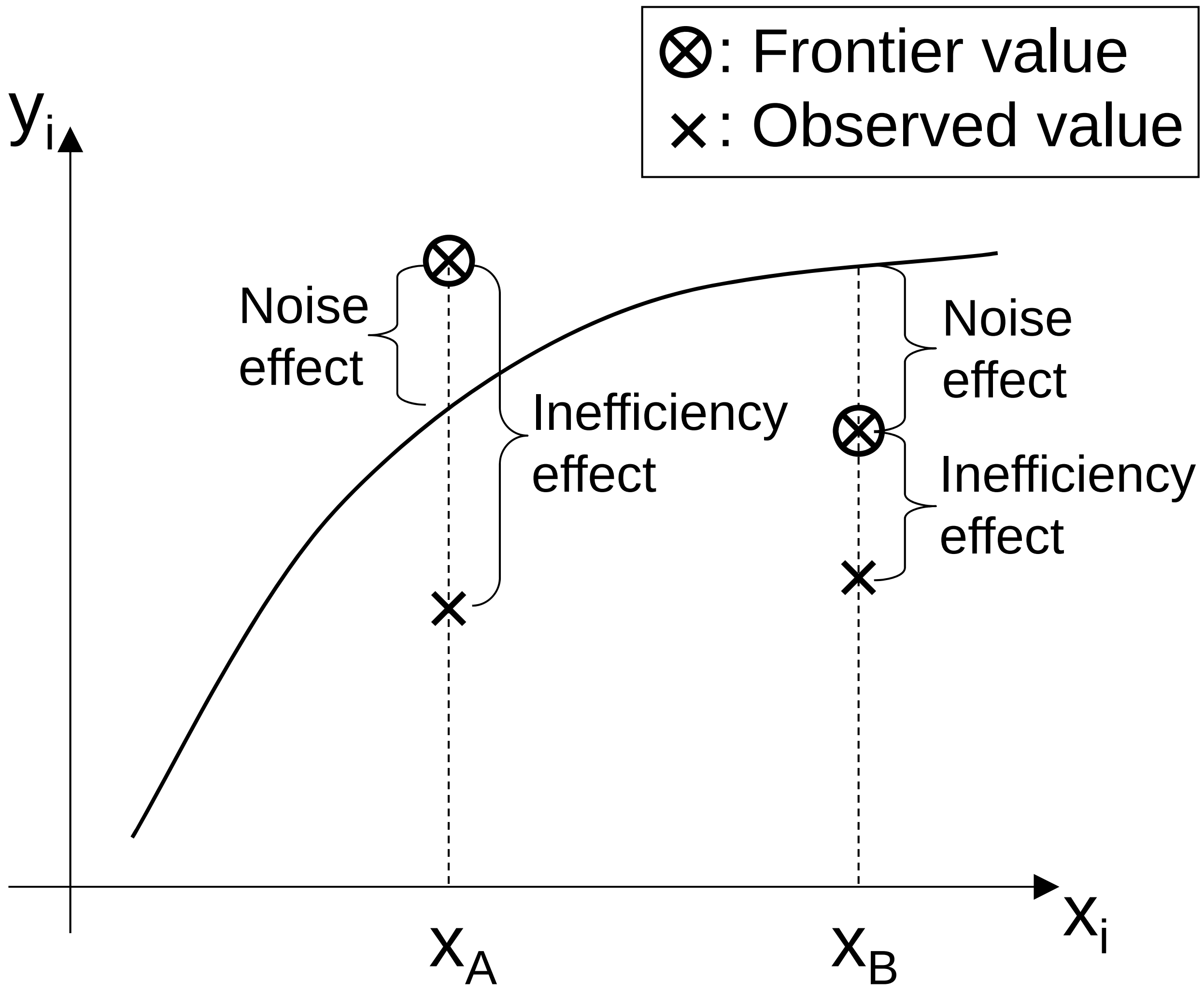
Non-negative random variable associated with inefficiency

- Stochastic production frontier

Statistical error

$$\boxed{\ln y_i = x_i^{'} \cdot \beta - e_i + \varepsilon_i \quad (i = 1, \ldots, N)}$$

Stochastic Frontier Analysis



SFA focuses on predicting the inefficiency effect