Master project option: Ricci curvatures for point clouds

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Curvature is a central concept in differential geometry, with a wide variety of applications in applied mathematics, from the study of thin elastic structures to general relativity. A more recent body of work also connects it to different aspects of purely data-driven problems, from predicting information bottlenecks in graphs [5] to quantifying robustness of classification boundaries with respect to adversarial attacks [3].

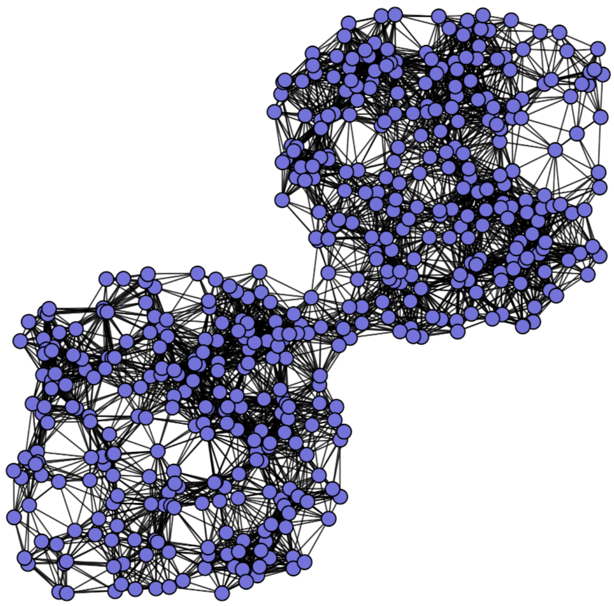

Figure 1: A graph with a bottleneck, detectable using intrinsic curvatures.

In this master thesis, you will focus on different types of intrinsic curvatures that could be applied to graphs constructed from point clouds sampled from the uniform distribution on a Riemannian manifold embedded in a higher-dimensional Euclidean space.

These include Forman Ricci curvature defined directly on the edges of the graph [4], spectral "low-frequency" analogues analogues defined through Weitzenböck-type formulas using the Bochner and Hodge Laplacians from spectral exterior calculus [1] or kernel approximations, and the "diffusion curvature" of [2].

First, we aim at a formal comparison of all these different definitions. Afterwards, we can take a few different directions:

- Numerical comparisons for generated data from well-understood manifolds of relatively low dimension (3,4,5...), to see how much these differ from each other and from the ground truth curvature, and their robustness to different kinds of perturbations.

- Working towards rigorous results for the discrete-to-continuum convergence of spectral and kernel curvatures, based on the extensive literature of convergence of point cloud Laplacians and their more recent spectral exterior calculus analogues.

References

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